

# WHEN DO SPACETIMES HAVE CONSTANT MEAN CURVATURE SLICES?

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ABSTRACT. Many results in mathematical relativity, including results for both the initial data problem and for the evolution problem, rely on the existence of a constant mean curvature (CMC) Cauchy surface in the underlying spacetime. However, it is known that some spacetimes have no CMC Cauchy surfaces (slices). This is an obstacle for many results and constructions with these types of spacetimes, and is particularly worrisome since it is not known whether spacetimes that do have CMC slices are in any sense generic. In this expository paper, we will discuss the known results about the existence (and non-existence) of CMC slices, examine the evidence for cases which are unknown, and make several conjectures concerning the existence of CMC slices and their generality.

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## 1. INTRODUCTION

As is well known, the problem of finding solutions to the Einstein Equations of general relativity can be split into finding initial data satisfying the constraint equations, and then evolving that data using the evolution equations. Both portions have their own interesting problems. What kinds of initial data are or are not possible? Is there a way to parameterize all initial data? Is the evolution problem stable? What happens near singularities, and what kinds of singularities can occur? Much progress has been made in recent decades on these kinds of problems. But looking at many of these papers, a common theme appears: *Constant mean curvature (CMC) initial data makes everything easier.*

In finding initial data in general relativity, the commonly used conformal method involves solving a system of coupled equations (the conformal constraint equations), in which the mean curvature function is one of the freely-specified parameters. If the chosen mean curvature is constant, then the equations decouple, making finding solutions much easier. Because of this, the CMC case of the initial data question is mostly understood. For instance, a parameterization of all closed CMC initial data has been found (see [11]). However, the conformal method in the non-CMC case is a completely different story, and is one of the main motivations for this study; we will come back to this shortly.

In the evolution problem in general relativity, there is some coordinate freedom; you may pick a lapse and shift arbitrarily. Obviously, some choices are better adapted to proving estimates than others. If you start with CMC initial data, there is a choice of lapse and shift that allow your time

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function to be the (constant) mean curvature of each time slice. This choice ends up being useful in proving various key estimates. For example, it is used in the recent resolution of the bounded  $L^2$  curvature conjecture by Klainerman, Rodnianski, and Szeftel [12], which states, essentially, that the evolution of the spacetime can continue as long as the  $L^2$  norm of the curvature remains bounded.

In both of these examples of results for the Einstein constraint and evolution equations, the authors make the assumption that the spacetimes they are considering have a CMC slice (Cauchy data). Unfortunately, not all spacetimes have CMC slices! This was first observed by Bartnik in [4], and vacuum examples were later found by Chruściel, Isenberg, and Pollack in [7]. This has serious implications for many of the results that have been obtained for both the Einstein constraint and evolution equations.

For those investigating initial data through study of the Einstein constraint equations, one major goal is the parameterization of all initial data. The conformal method works well as this parameterization for CMC initial data but, unfortunately, it fails for non-CMC data. The solution space of the conformal constraint equations for far-from-CMC data exhibit folds and blowup in unexpected places. This has been shown analytically in some cases [14, 15], and numerically in others [8]. The behavior of the conformal constraint equations in the far-from-CMC regime is very complicated. However, if every spacetime had a CMC Cauchy surface (slice), while the conformal method would not parameterize all initial data, it would at least parameterize initial data for every spacetime. Unfortunately, not all spacetimes have CMC slices, but a similar statement could be made as long as spacetimes with CMC slices are generic.

Many of the most important conjectures in mathematical relativity ask whether generic spacetimes have (or don't have) certain properties or features, such as Cauchy horizons. However, in many papers about the evolution problem, assuming the existence of a CMC slice is a key first step in the analysis. While this may be more a limitation of methods rather than an actual obstruction, it is still worrying. These results (such as the  $L^2$  curvature theorem) cannot be used to prove anything about generic spacetimes, unless it is known that a generic spacetime has a CMC slice. That is currently unknown. Additionally, is it possible that the spacetimes without CMC slices are an obstruction that makes proving these results more difficult? The main concern is that it is unknown how general spacetimes without CMC slices are. While there are a few examples, they are very special examples, with high levels of symmetry. Is there an open set of such spacetimes? Are they ubiquitous, or very special?

In this expository paper, we will discuss known results about the existence and non-existence of constant mean curvature slices in spacetimes, possible directions for investigation, and make a few conjectures.

## 2. TECHNICAL BACKGROUND

A Lorentzian manifold  $\mathcal{V}$  is a *cosmological spacetime* if it is globally hyperbolic with compact Cauchy surfaces and satisfies the *timelike convergence condition*,

$$(1) \quad R_{\mu\nu}T^\mu T^\nu \geq 0, \text{ for every timelike vector } T.$$

If the spacetime also obeys the Einstein equations, this is equivalent to the strong energy condition. Many results included will not require that  $\mathcal{V}$  satisfies the Einstein equations, though we are predominately interested in their solutions. By *slice*, we will always mean a compact Cauchy surface.

Intuitively, the mean curvature gives the expansion and contraction rate of the universe. We will use the sign convention that  $\text{tr}k = H < 0$  means the universe is contracting to the future. It is well known [10] that if  $H < 0$  on a slice, then all future timelike geodesics must be incomplete (in the globally hyperbolic development), with a uniform upper bound on their proper length of  $3/ - H$ .

There are a number of well known, useful facts about CMC slices in cosmological spacetimes, when they exist (see [4, 13]). For instance, if a CMC slice exists in  $\mathcal{V}$ , with  $H \neq 0$ , it is necessarily unique. For maximal slices, with  $H \equiv 0$ , we can only say that the spacetime is static, with a timelike Killing field  $T$  with  $R_{\mu\nu}T^\mu T^\nu = 0$ .

When a spacetime has a CMC slice, there is a foliation, at least locally, by CMC slices. The mean curvatures vary along the foliation monotonically, except, sometimes, when  $H \equiv 0$ . In the maximal case, nearby slices may also be maximal, but if they are, the spacetime is static. A major question (see, for instance, [16]) is whether this foliation covers the spacetime, and whether they necessarily achieve all possible mean curvatures.<sup>1</sup>

For any two slices with mean curvatures  $H_1$  and  $H_2$ , and one in the future of the other, then for any mean curvature  $H$  satisfying  $\sup H_1 \leq H \leq \inf H_2$ , then there is a slice with mean curvature  $H$  in the spacetime, between the other slices.

One common use for this result is that, if  $\mathcal{V}$  has CMC slices of mean curvature  $H_1 \leq H_2$ , then the region between the slices is foliated by CMC slices, with mean curvatures monotonically increasing along the foliation from  $H_1$  to  $H_2$ .

It will also be useful to define a spacetime ray and line.

**Definition 2.1.** A future (resp. past) *ray* is a future (resp. past) timelike path with infinite affine length.

**Definition 2.2.** A *line* is a a timelike *geodesic* with infinite affine length to the future *and* past, which is also *globally* maximizing in distance.

The earlier mentioned result, that if there is a slice with  $H < 0$ , then timelike geodesics have an upper bound on length to the future, shows that the existence of a CMC implies the nonexistence of rays, either to the past or future, depending on the sign. On the other hand, the importance of lines is shown by the following theorem from [4].

**Theorem 2.3.** *A globally hyperbolic spacetime satisfying the timelike convergence criterion (1) and admitting a line is a metric product. Thus, it has maximal slices.*

### 3. WHICH SPACETIMES HAVE CMC SLICES?

The most basic question to ask is, “Under what conditions does a spacetime have or not have CMC slices?”

Since CMC slices are unique, it is easy to cut enough out of a spacetime to create a spacetime without a CMC slice. We are not concerned with such examples. In order to avoid that, we will consider only maximal globally hyperbolic developments (MGHDs).

Unfortunately, since work on barrier methods in the 80’s, by Bartnik and others, there are relatively few papers that deal with the existence of a CMC slice. (Most papers assume there is one, then investigate the foliation by CMC slices.) Various barrier methods are well developed. The

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<sup>1</sup>If the topology of the slices do not allow a metric with positive scalar curvature, the spacetime cannot have a maximal slice. Thus, all allowable mean curvatures would be  $(-\infty, 0)$  or  $(0, \infty)$ .

barriers are essentially used to guarantee that the slices avoid the singularities at future or past infinity.

One type of barrier was used in [4]. In that paper, he showed that if  $\mathcal{V} - I(p)$  is compact for some point  $p \in \mathcal{V}$ , then there is a CMC slice, passing through  $p$ . The barrier in the proof is  $\partial I(p)$ , which is also compact. He uses this boundary to show that there are CMC slices through  $p$ , though singular at  $p$ , with any desired mean curvature. He then shows that one of these slices must in fact be smooth.

The condition that  $\mathcal{V} - I(p)$  is compact is an interesting condition. It roughly states that an observer at  $p$  could observe all events that happened in the universe, sufficiently far in the past. There is no “hidden” portion of the universe. A similar statement can be said about the future. However, it is not a necessary condition. Spherical FRLW spacetimes don’t satisfy it (light rays emitted from the big bang don’t travel across the universe before recollapse, and so there can be no such  $p$ ), but they still have CMC slices.

A more common kind of barrier is a slice with a certain mean curvature. As discussed in Section 2, if we can find two slices, one in the future of the other, and with mean curvatures  $\sup H_1 \leq \inf H_2$ , then there is a CMC slice between them.

This immediately shows that a spacetime with a crushing singularity has CMC slices. A (future) *crushing singularity* means that there is a sequence of slices  $\Sigma_i$  approaching the future edge of the MGH, with mean curvatures  $H_i$  satisfying  $\sup_{\Sigma_i} H_i \rightarrow -\infty$ . If you have a crushing singularity, pick any slice  $\Sigma$  as one barrier, then use the crushing singularity to find a slice  $\Sigma'$  with  $\sup_{\Sigma'} H' \leq \inf_{\Sigma} H$ . A CMC slice is then between them.

It is then useful to find other conditions which guarantee that your future singularity is crushing. One such condition is the strong curvature singularity.

**Definition 3.1.** A timelike geodesic  $\lambda(t)$  terminates in a *strong curvature singularity* at affine parameter value  $t_0$  if the following holds: Let  $\mu(t)$  be the three-form on the normal space to  $\lambda'(t)$  determined by any three linearly independent vorticity-free Jacobi fields along  $\lambda(t)$ . If  $\mu(t)$  vanishes for at most finitely many  $t$  is some neighborhood  $[t_1, t_0)$  of  $t_0$ , then we require  $\lim_{t \rightarrow t_0} \mu(t) = 0$ .

For cosmological spacetimes, if  $\mu(t)$  vanishes finitely many times, the limit exists. The basic equation is

$$(2) \quad \frac{d^2}{dt^2} \mu(t)^{1/3} + \frac{1}{3} (R_{ab} v^a v^b + 2\sigma^2) \mu^{1/3} = 0,$$

and, importantly, note that the second term is positive, which makes  $\mu$  behave well. Using this equation, it can be shown that any global strong curvature singularity is also a crushing singularity.

**Theorem 3.2.** [13, Thm 6] *For a cosmological spacetime, if there exists a Cauchy surface from which all orthogonal (future or past) timelike geodesics end in a strong curvature singularity, then that singularity is crushing, and so there is a CMC slice.*

Another theorem in that same paper gives another condition guaranteeing that a singularity is crushing. It has to do with the  $c$ -boundary. The future  $c$ -boundary is essentially constructed by associating each point  $p$  of the spacetime with its past  $I^-(p)$ . The future boundary is then defined as the sets  $I^-(\gamma)$ , where  $\gamma$  is an inextendible future directed timelike path. (The technical details for these “indecomposable pasts” can be complicated, but we won’t need them in this paper.)

**Theorem 3.3.** [13, Thm 6] *Suppose a cosmological spacetime with global future singularity has  $c$ -boundary consisting of a single “point,” i.e., that all inextendible future timelike paths have the*

same past (and thus, that past is all of the spacetime.) Then the singularity is crushing, and so a CMC slice exists.

Those are the most general results that we are aware of for proving that a CMC slice exists.

Let's now turn our attention to the known examples of spacetimes without CMC slices. Unfortunately, only a few examples are known.

Though this paper is predominantly focused on spatially compact spacetimes, we should briefly mention results for spatially asymptotically Euclidean manifolds. In these spacetimes, an CMC asymptotically Euclidean Cauchy surface can only have  $H \equiv 0$ , i.e., be maximal. If the spacetime obeys the weak energy condition, then the Cauchy surface must have a metric of positive scalar.

However, just as in the compact case, there are topological obstructions to this. An explicit example was found by Brill in [5], but it is now known [9] that any asymptotically Euclidean manifold, if it can be one point compactified (one point for each end) to a manifold not allowing a metric of positive scalar curvature, itself cannot have an asymptotically Euclidean metric of positive scalar curvature. This gives well-understood topological restrictions to the existence of maximal Cauchy surfaces in spatially asymptotically Euclidean spacetimes.

For spatially compact spacetimes, the original, explicit example comes from Bartnik's paper [4]. To construct his example, take the maximally extended Schwarzschild spacetime. Then, on each of the ends, cut off each end and attach a torus with a homogeneous dust in it. Bartnik uses a particular model (Tolman-Bondi) that allows gluing between these regions in such a way that the spacetime evolves nicely and satisfies the strong energy condition.

Bartnik proves that this spacetime has no CMC slices in two ways.

The first way is by symmetry. The spacetime has a "time inversion" symmetry. In particular, if there is a non-maximal CMC slice, then by this symmetry there is one with the same mean curvature but opposite sign.

In between these slices, you can foliate the spacetime with CMC slices, and thus find a maximal slice. But the spacetime satisfies the weak energy condition, and so the maximal slice has non-negative scalar curvature by the constraint equations. This contradicts the topology of the slices,  $T^3 \# T^3$ .

This proof method is simple, and shows that the topology of the slices may be important in proving more general results. However, symmetry is obviously not a generic property, and so this proof seems limited to very symmetric data, a serious limitation.

However, it also allowed the development of a family of examples in [7]. In this paper, they take  $T^3$  vacuum solutions, then glue them (using IMP gluing) symmetrically. The important addition is that these examples are vacuum, and so the dust from Bartnik's example was not necessary.<sup>2</sup> However, it is still hard to generalize these examples due to the symmetry requirement.

Bartnik's second proof uses timelike path incompleteness.

**Theorem 3.4** ([4]). *If a globally hyperbolic, cosmological spacetime  $\mathcal{V}$  has a future ray and a past ray, but no line, then  $\mathcal{V}$  has no CMC Cauchy surfaces.*

*Proof.* Suppose  $\mathcal{V}$  did have a CMC slice, with  $H \neq 0$ . Then by [10, pg 274], all future (or past) timelike paths have finite length (depending on the sign of  $H$ .) This is a contradiction.

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<sup>2</sup>It is known that dust can cause shell-crossing singularities (though there are none in Bartnik's example), and so dust is sometimes avoided.

If  $H \equiv 0$ , the spacetime is locally foliated either by CMC slices with nonzero  $H$  (in which case the first paragraph applies), or it is foliated by maximal slices. If this maximal foliation covers the spacetime, the spacetime is static and thus contains a line. This is a contradiction.  $\square$

The explicit example constructed by Bartnik has rays (complete timelike paths to the past or future), because the Schwarzschild spacetime does. However, it clearly has no line (and, in fact, no complete timelike geodesics at all), and so cannot have a CMC slice.

As this second proof doesn't use symmetry, this approach seems a more likely candidate for generalization. Indeed, our main conjecture is that this condition is both necessary and sufficient.

**Conjecture 3.5.** A globally hyperbolic, cosmological spacetime has no CMC Cauchy surfaces if and only if it contains a future and a past ray, but no line.

To see why this conjecture is reasonable, recall Theorem 2.3, which says that if a spacetime has a line, then your spacetime is a metric product, and thus have maximal slices. The difficult part of the conjecture is then to show the following:

**Conjecture 3.6.** If, in a cosmological spacetime, every future (or past) timelike path is incomplete (i.e., of finite length), then there exists a CMC slice.

In this formulation the conjecture essentially says that, if you have a global singularity, then you have a CMC slice.

For certain kinds of singularities, such as the crushing and strong curvature singularities we discussed earlier, we know this conjecture holds. Unfortunately, not all singularities are necessarily of the crushing type. For instance, in [16, pg 3594], Rendall constructs an example where a dust shell-crossing singularity causes a CMC foliation to stop at a finite mean curvature, and so the singularity is not crushing. Of course, this example uses dust, which is known to cause problems, and still has CMC slices. We are aware of no other cosmological MGHs, ending in a global singularity, which do not have a crushing singularity.

Both of Bartnik's proofs in fact prove something stronger than that the spacetime has no CMC slices. They also show that the spacetimes have no slices of constant signed mean curvature. This observation leads to a conjecture.

**Conjecture 3.7.** A cosmological spacetime has a CMC slice if and only if it has a slice of constant signed mean curvature.

Since a slice of constant (non-zero) signed mean curvature implies that there can be no rays, the main conjecture 3.6 would imply this conjecture. However, it provides an interesting intuitive picture for spacetimes without CMC slices.

If this conjecture is true, then every slice must have a region with positive mean curvature and a region with negative mean curvature. Because of this, there appears to be one region of the spacetime that expands for all time, and another that expands to the past for all time. (This also lines up with the existence of rays, as in the main conjecture 3.6.) If these regions were causally linked, you might expect to find a line going from the one region to the other, which would lead to a contradiction. Thus the spacetime has (at least) two regions separated by some sort of "wormhole," as in Bartnik's example, with one region expanding, the other contracting.

The one implication of the conjecture 3.7 is obvious. The other makes sense heuristically. If  $\mathcal{V}$  has a slice of constant signed (non-maximal) mean curvature, there is a global singularity to the future or past. By the previous conjecture, there must be a CMC slice.

More directly, suppose that  $\mathcal{V}$  has a slice of constant signed (non-maximal) mean curvature. It is possible to evolve the slice by mean curvature flow. On a Lorentzian manifold, the mean curvature flow tends to expand slices, and so the slice will evolve into the expanding universe direction, i.e., towards  $H \equiv 0$ . As the universe is expanding, you would not expect singularities to appear before you reach the point where the new mean curvature is globally closer to zero than the original mean curvature.<sup>3</sup> Using the standard interpolation result, there is a slice of constant mean curvature between them.

The most useful condition for showing the existence or nonexistence of a CMC slice would refer only to the initial data. For instance, if a set of vacuum initial data is of Yamabe class  $Y^+$ , the closed universe recollapse conjecture [3] says that the spacetime will begin and end in a global singularity. If this conjecture is true, and the main conjecture 3.6 is true, then every Yamabe positive vacuum initial data set leads to a spacetime with CMC slices.

Another idea is to try to find conditions on a region of initial data such that that region, when evolved, will contain rays. For instance, like in Bartnik's example, if there is a region of the initial data with positive mean curvature, and it is separated from other regions by a black-hole-type region, then it seems reasonable that this region should expand for all time, and thus have rays. Then, if an initial data set had one region expanding to the future in this way, and one to the past, then, by Theorem 3.4, the spacetime cannot have any CMC slices. This condition would be especially interesting if it were stable under perturbations, which it seems like it should be.

The difficulty in any attempt to find conditions on initial data is that proving that there are or are not CMC slices in the spacetime requires asking questions about the long time behavior of the spacetime. Results of this nature tend to be difficult to prove, and many of the best results require the existence of a CMC slice. This difficulty is the reason that the only known condition on initial data is the very restrictive time-antisymmetric condition used earlier.

#### 4. ARE CMC SLICES GENERIC?

The most important question about CMC slices is their genericity. If the existence of CMC slices is generic, then the special examples where they do not exist can mostly be ignored, and CMC slicing can be used for proving generic properties. Unfortunately, to our knowledge, there is no evidence that this is the case, other than the weak evidence that all known examples are all very special, all on  $T^3 \# T^3$  with time-antisymmetry.

In any question of genericity, it is vital to choose a useful and meaningful topology. In questions about spacetimes, there are several choices that could be made.

The simplest choice would be a global  $C^{2,\alpha}$  topology. Using this topology, Choquet-Bruhat [6] showed that spacetimes with CMC slices form an open set. In particular, she proves that perturbations in a neighborhood of the evolution of a CMC slice always has another CMC slice. This is done using the linearization of the mean curvature operator.

However, in this topology, the existence of CMC slices is not generic! Due to Theorem 3.4, if there are timelike rays, but no timelike line, then the spacetime does not have a CMC slice.

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<sup>3</sup>Indeed, this can almost be carried out. Using basic estimates, it is not hard to show that as long as all timelike geodesics exist for a certain explicit amount of proper time into the expanding direction (based *only* on the starting mean curvature), then you can evolve the slice long enough to get the desired slice with small mean curvature. Unfortunately, in trying to show that you can always evolve long enough to find such a slice, you need an estimate of certain quantities. The type of estimate that would seem to work is exactly the kind of estimate that CMC slices are used for in evolution problem papers—a relationship between the second fundamental form and the lapse. Thus the problem becomes somewhat circular.

However, the unboundedness of rays is stable under a global  $C^{2,\alpha}$  perturbation. Also, if there were a line, Theorem 2.3 says the spacetime would be static. However, Bartnik's example is not  $C^{2,\alpha}$  close to a static spacetime, and so a small enough  $C^{2,\alpha}$  perturbation could not create a line. Together, this means that every small, global  $C^{2,\alpha}$  perturbation of Bartnik's example would still have no CMC slices.

We would be among the first to admit, though, that the global  $C^{2,\alpha}$  topology, along with any other global topology, is probably not the right one for this problem. For instance, since we are dealing only with MGHs, spacetimes end in singularities or exist for all time. In either case, something is unbounded. Thus,  $C^{2,\alpha}$  close is extremely restrictive.

A related problem is that there are one parameter families of initial data which are not close to each other in the  $C^{2,\alpha}$  norm. For example, initial data for the flat toroidal, static universe can be perturbed into initial data which evolve into spacetimes with either global future or global past singularities.

It seems that a topology based on initial data is more reasonable for this problem. We propose to use the  $C^{2,\alpha}$  topology on initial data sets  $(M, g, K)$ .

Using this topology, the existence of CMC slices is still an open condition. Consider initial data evolving into a spacetime with at least one CMC slice. That CMC slice is within some amount of coordinate time  $t_0$  from the initial slice. For any initial data sufficiently close to the original data, the spacetimes are  $C^{2,\alpha}$  close within  $t_0$  of the initial data. Again using Choquet-Bruhat's argument in [6], the nearby spacetime also has a CMC slice, of the same mean curvature.

Unfortunately, as with the global  $C^{2,\alpha}$  topology, it appears that the existence of CMC slices is not a generic condition.

**Conjecture 4.1.** There is an open set of initial data (in the  $C^{2,\alpha}$  norm) such that the associated spacetimes do not have CMC slices.

Consider Bartnik's example that we discussed earlier. To remind the reader, his spacetime is two toroidal universes, (one expanding to the future, the other to the past,) separated by a Schwarzschild bridge. If the existence of CMC slices was a dense condition, a generic perturbation of initial data for Bartnik's example should evolve into a spacetime with CMC slices.

This perturbed data could not lead to a static spacetime, and so it has a non-maximal CMC slice. Thus, all future (or past) geodesics are incomplete. In other words, a generic perturbation causes at least one of the expanding toroidal universes to collapse. But a small perturbation shouldn't cause global collapse. To make this a bit more precise, we make the following conjecture.

**Conjecture 4.2.** If a compact, vacuum initial data set evolves into a non-static spacetime such that there is a family of future rays intersecting the initial Cauchy surface in an open set, then any small  $C^{2,\alpha}$  perturbation of the initial data evolves into a spacetime with at least one future ray.

This is not the same as stability of the spacetime. For instance, the perturbation could coalesce into a black hole type region, but as long as it didn't cause the global collapse of the expanding region, the conjecture would still hold.

If this conjecture were true for initial data with dust, Bartnik's example would immediately imply that the existence of CMC slices is not generic. Similarly, the examples of Chruściel, Isenberg, and Pollack, if they contain a ray, would also imply that the existence of CMC slices is not generic.

Another heuristic example can be found by gluing together two stable 2-tori. In two papers, [1, 2], Andersson and Moncrief give vacuum, hyperbolic FLRW models with the metric  $-d\rho^2 + \rho^2\gamma$ , where  $\gamma$  is the hyperbolic metric on the 2-torus slice. Importantly, they prove these spacetimes



are stable attractors of nearby initial data, at least in the expanding direction. If we take two of the same constant- $\rho$  slices, but change the sign of the second fundamental form  $k$  on the second one, we can do the same gluing they used in [7] in order to glue these together anti-symmetrically in  $k$  and symmetrically in the metric. Then, as with their examples, the evolution can have no CMC slices due to the symmetry and bad topology.

Of course, these examples are really just a special case of the examples in [7]. The advantage these have is that the base hyperbolic space we are using is known to be stable. That means, after the gluing, in the domain of dependence (in the expanding direction) away from the gluing region, the spacetime is close to, and in fact converging toward, the original FLRW. If this domain existed for all time under any perturbation, the new spacetime would have infinite length timelike paths, and thus no CMC slices. That would show that the set of initial data that evolve to have CMC slices is *not* dense!

Unfortunately, it is unclear whether this domain of dependence lasts for all time. In the original FLRW spacetime, it is straightforward to check that the expansion rate of the universe is just barely too slow to guarantee that. (If the  $\rho^2$  were replaced by  $\rho^2 \ln^3(\rho)$ , for instance, it would work.)

However, perturbations of this glued spacetime likely still have no CMC slices. Heuristically, the wormhole bridge connecting the two glued tori should act like the Schwarzschild bridge in Bartnik's example. Since it is a black-hole-like region, we would expect it to stay small, only lightly affecting the spacetime outside its immediate neighborhood. If this were true, the spacetime would not recollapse in the expanding direction, and so all perturbations would lead to spacetimes without CMC slices.

It is, of course, possible that spacetimes with CMC slices are generic. If that were true, for any initial data set  $S$  leading to a spacetime without CMC slices, there are sets of initial data  $S_i$ , leading to spacetimes *with* CMC slices, converging to the initial data in the  $C^{2,\alpha}$  topology. Since the corresponding spacetimes  $\mathcal{V}_i$  will then converge on any compact interval of time (of the spacetime  $\mathcal{V}$  evolved from  $S$ ), if the CMC slices of the  $\mathcal{V}_i$  were well behaved, then the boundary spacetime  $\mathcal{V}$  would also contain a CMC slice. Thus, the CMC slices of the  $\mathcal{V}_i$  cannot be well behaved.

There are three ways the convergence of CMC slices could fail. The first is that for each  $H$ , the CMC slice in  $\mathcal{V}_i$  of mean curvature  $H$  could “run off” to infinity, so that, in the limit, no compact interval of time could contain the slices. The second is that these slices may become null in the limit. For the third, it is possible that, for any  $H$ , for  $i$  large enough,  $\mathcal{V}_i$  does not have a CMC slice of mean curvature  $H$ . In other words, the foliation by CMC slices in  $\mathcal{V}_i$  covers a vanishing interval of the possible mean curvatures.

In any of these cases, the boundary spacetime  $\mathcal{V}$  should have slices of arbitrarily near constant mean curvature. If the converging slices are non-maximal, the slices in  $\mathcal{V}$  should be near-CMC in the traditional sense of  $\|\nabla H\|$  being small compared to  $\|H\|$ . If the converging slices are maximal (or approach maximal), as they would be in Conjecture 3.7 were true, the slices in  $\mathcal{V}$  would be near-CMC but perhaps only in the sense that the average of  $H$  could be made arbitrarily small.

Thus, heuristically, the existence of CMC slices is a generic property if and only if every spacetime has near-CMC slices. Thus, one way to check whether we should expect this property to be generic is to check whether the known examples have arbitrarily near-CMC slices. If they do not, we would expect that the existence of CMC slices is not generic. If they do, we cannot, however, conclude the opposite.

Let us mention that this heuristic argument has consequences for the initial data problem. The conformal method, as mentioned in the introduction, parameterizes all CMC initial data. While

it fails for far-from-CMC data, it still behaves well for near-CMC data, including existence and uniqueness. If it were true that all spacetimes have near-CMC slices, the conformal method may still be able to provide a reasonable parameterization of initial data for all spacetimes.

While genericity is the main question, it is not even known whether spacetimes with CMC slices and those without are even in the same connected component. To prove that they aren't, one could try to show that the no-CMC-slice condition is open, or that the CMC-slice condition is closed. Both of those methods, if our main Conjecture 3.6 is true, are questions about long term stability of the spacetime: If a spacetime has a ray, do all nearby spacetimes also have a ray?

At first glance it may seem like a spacetime with a single (future) ray could be perturbed to a spacetime without a ray. However, do such spacetimes even exist? By conjecture 3.7, which says that spacetimes without CMC slices will have no slices of constant sign, we expect that a spacetime without CMC slices will be expanding in some region. An expanding region should have an open set of rays.

Additionally, an expanding region should be stable. The boundary example of the static universe is unstable, and cannot occur since there is a region that is expanding to the past as well. Any sufficiently small perturbation of the expanding region, as we've argued before, should not cause the expanding region to collapse. This would suggest that having no CMC slices is also an open condition, and thus forms a disconnected component of initial data.

One could also try to show connectedness direction. The most obvious way is to take the initial data for an example with no CMC slices, and treat it as seed data for the conformal constraint equations. Then one can make a path of seed data from that given data to one with constant mean curvature. The difficulty is that one must solve the conformal constraint equations for each set of seed data in order to find initial data for a spacetime. Unfortunately, it is now well established (see [8]) that for far-from-CMC seed data, which we, of necessity, would have to work with, the conformal constraint equations are very complicated, and it is unclear that they have solutions for that seed data.

## 5. FINAL COMMENTS

Constant mean curvature Cauchy surfaces are useful and much easier to work with than their non-CMC counterparts. This is true both for initial data and for the evolution of the data. Unfortunately, while much is known about foliations by CMC slices, *given* a starting CMC slice, less has been written about whether or not spacetimes have CMC slices at all.

Importantly, many results assume that the spacetime has a CMC slice. Unfortunately, it is currently unknown whether or not that assumption is true for a generic spacetime. Our conjecture is that it is *not* a generic condition. Since many of the most important conjectures in mathematical relativity concern generic spacetimes, this means that results that assume the existence of a CMC slice are not applicable to those problems, unless the assumption of the existence of a CMC slice can somehow be removed.

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