GEOMETRIC VARIATIONAL CRIMES: HILBERT COMPLEXES, FINITE ELEMENT EXTERIOR CALCULUS, AND PROBLEMS ON HYPERSURFACES

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Abstract. A recent paper of Arnold, Falk, and Winther [*Bull. Amer. Math. Soc.* **47** (2010), 281–354] showed that a large class of mixed finite element methods can be formulated naturally on Hilbert complexes, where using a Galerkin-like approach, one solves a variational problem on a finite-dimensional subcomplex. In a seemingly unrelated research direction, Dziuk [*Lecture Notes in Math.*, vol. 1357 (1988), 142–155] analyzed a class of nodal finite elements for the Laplace–Beltrami equation on smooth 2-surfaces approximated by a piecewise-linear triangulation; Demlow later extended this analysis [*SIAM J. Numer. Anal.*, **47** (2009), 805–827] to 3-surfaces, as well as to higher-order surface approximation. In this article, we bring these lines of research together, first developing a framework for the analysis of variational crimes in abstract Hilbert complexes, and then applying this abstract framework to the setting of finite element exterior calculus on hypersurfaces. Our framework extends the work of Arnold, Falk, and Winther to problems that violate their subcomplex assumption, allowing for the extension of finite element exterior calculus to approximate domains, most notably the Hodge–de Rham complex on approximate manifolds. As an application of the latter, we recover Dziuk's and Demlow's *a priori* estimates for 2- and 3-surfaces, demonstrating that surface finite element methods can be analyzed completely within this abstract framework. Moreover, our results generalize these earlier estimates dramatically, extending them from nodal finite elements for Laplace–Beltrami to mixed finite elements for the Hodge Laplacian, and from 2- and 3-dimensional hypersurfaces to those of arbitrary dimension. By developing this analytical framework using a combination of general tools from differential geometry and functional analysis, we are led to a more geometric analysis of surface finite element methods, whereby the main results become more transparent.

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Contents

1. Introduction

The aim of this paper is to bring together three distinct ideas that have influenced, in separate ways, the development and analysis of geometric finite element methods for elliptic partial differential equations.

The first idea is that of a *variational crime*. Suppose we have a variational problem of the form: Find $u \in V$ such that

$$B(u,v) = F(v), \quad \forall v \in V, \tag{1}$$

where V is a Hilbert space, $B\colon V \times V \to \mathbb{R}$ is a bounded, coercive bilinear form, and $F \in V^*$ is a bounded linear functional. If $V_h \subset V$ is a subspace (usually finite-dimensional), then one can obtain an approximate solution by solving the Galerkin variational problem: Find $u_h \in V_h$ such that

$$B(u_h, v) = F(v), \quad \forall v \in V_h.$$

This is the typical abstract setting for finite element methods. However, for many problems of interest, especially finite element methods on surfaces or on domains with curved boundaries, one cannot efficiently compute the bilinear form $B(\cdot,\cdot)$ or the functional $F(\cdot)$ on a subspace of V. Instead, one must take an approximating space $V_h \not\subset V$, along with an approximate bilinear form $B_h\colon V_h \times V_h \to \mathbb{R}$ and functional $F_h \in V_h^*$, and formulate the generalized Galerkin variational problem: Find $u_h \in V_h$ such that

$$B_h(u_h, v) = F_h(v), \quad \forall v \in V_h. \tag{2}$$

Such modifications to the original variational problem are called "variational crimes." There is a well-understood framework for the analysis of a large class of variational crimes, represented by the *Strang lemmas* [7]: for instance, the first and second Strang lemmas allow for the complete analysis of numerical quadrature, the use of geometric modeling technology such as isoparametric elements, and many other examples of variational crimes.

The emergence of *surface finite elements* represents a second distinct idea that has influenced the development of geometric finite element methods. The analysis of surface finite element methods, which by construction are "criminal" methods, has required a more sophisticated approach that exploits the specific nature of the crime in order to obtain a satisfactory error analysis; this custom-tailored analysis contrasts with the more general approach given by the Strang lemmas. The surface finite element research area was effectively initiated with the 1988 article of Dziuk [16], although there is related work appearing about ten years earlier by Nédélec [26]. While there was some activity in the area during the 1990s (cf. [17, 11]), beginning in 2001 there was a tremendous expansion of research in the general area of surface finite element methods, with many applications arising in material science, biology, and astrophysics; examples include [21, 10, 12, 13, 19, 18, 15, 14].

The third distinct idea that has had a major influence on the development of geometric methods is that of *mixed finite elements*, whose early success in areas such as computational electromagnetics was later found to have surprising connections with the calculus of exterior differential forms, including de Rham cohomology and Hodge theory [6, 27, 28, 20]. This has culminated, very recently, in the powerful theory of *finite element exterior calculus* developed by Arnold, Falk, and Winther [2, 3]. A key insight of the latter work, from a functional-analytic point of view, is that a mixed variational problem can be posed on a *Hilbert complex*: a differential complex of Hilbert spaces, in the sense of Brüning and Lesch [8]. Galerkin-type mixed methods are then obtained by solving the variational problem on a finite-dimensional subcomplex.

In this article, we bring these lines of research together, first developing a framework for the analysis of variational crimes in abstract Hilbert complexes, and then applying this abstract framework to the setting of finite element exterior calculus on hypersurfaces. Our framework extends the work of Arnold, Falk, and Winther [3] to problems that violate their subcomplex assumption, allowing for the extension of finite element exterior calculus to approximate domains, most notably the Hodge–de Rham complex on approximate manifolds. As an application of the latter, we recover Dziuk's [16] and Demlow's [14] *a priori* estimates for 2- and 3-surfaces, demonstrating that surface finite element methods can be analyzed completely within this abstract framework. Moreover, our results generalize these earlier estimates dramatically, extending them from nodal finite elements for Laplace–Beltrami to mixed finite elements for the Hodge Laplacian, and from 2- and 3-dimensional hypersurfaces to those of arbitrary dimension. By developing this analytical framework using a combination of general tools from differential geometry and functional analysis, we are led to a more geometric analysis of surface finite element methods, whereby the main results become more transparent.

The remainder of the article is organized as follows. In Section 2, we review the abstract framework of Hilbert complexes, which plays a central role in the work of Arnold, Falk, and Winther [3] on finite element exterior calculus. This includes a brief introduction to Hilbert complexes and their morphisms, domain complexes, Hodge decomposition, the Poincaré inequality, the Hodge Laplacian, mixed variational problems, and approximation using Hilbert subcomplexes. In Section 3, we consider the approximation of a Hilbert complex by a second complex, related to the first complex through an injective morphism rather than through subcomplex inclusion. Since this morphism is not necessarily unitary (i.e., inner-product preserving), this allows the approximating complex to have a different inner product, which only approximates that of the original complex. We develop some basic results for the pair of complexes and the maps between them, and then prove error estimates for generalized Galerkin-type approximations of solutions to variational problems using the approximating complex; these estimates generalize the results of Arnold, Falk, and Winther [3] to "external" approximations. Our results may be viewed as establishing *Strang-type lemmas* for approximating variational problems in Hilbert complexes. Finally, in Section 4, we apply the framework developed in Section 3 to the Hodge–de Rham complex of differential forms on a compact, oriented Riemannian manifold. We first review Hodge–de Rham theory, and then consider a pair of Riemannian manifolds related by diffeomorphisms, establishing estimates for the maps needed to apply the generalized Hilbert complex approximation framework. After reviewing the concept of a tubular neighborhood, we then consider the specific case of Euclidean hypersurfaces. We subsequently show how the results of the previous sections recover the analysis framework and *a priori* estimates of Dziuk [16], Demlow and Dziuk [15], Demlow [14], and moreover extend their results from scalar functions on 2- and 3-surfaces to general k-forms on arbitrary dimensional hypersurfaces. We also indicate how our results generalize the *a priori* estimates of Dziuk [16], Demlow [14] from nodal finite element methods for the Laplace–Beltrami operator to mixed finite element methods for the Hodge Laplacian.

2. Review of Hilbert complexes

In this section, we quickly review the abstract framework of Hilbert complexes, which forms the heart of the analysis in Arnold, Falk, and Winther [3] for mixed finite element methods. Just as the space of L^2 functions is a prototypical example of a Hilbert space, the prototypical example of a Hilbert complex to keep in mind is the L^2-de Rham complex of differential forms. (This example will be discussed at greater length in Section 4.) After stating the basic definitions, we will summarize some of the key results from Arnold, Falk, and Winther [3] on mixed variational problems and their numerical approximation using Hilbert subcomplexes. The interested reader may also refer to Brüning and Lesch [8] for a comprehensive treatment of Hilbert complexes from the viewpoint of functional analysis.

2.1. **Basic definitions.** Let us introduce the basic objects of study, Hilbert complexes, and their morphisms.

Definition 2.1. A *Hilbert complex* (W, d) consists of a sequence of Hilbert spaces W^k, along with closed, densely-defined linear maps $\mathrm{d}^k \colon V^k \subset W^k \to V^{k+1} \subset W^{k+1}$, possibly unbounded, such that $\mathrm{d}^k \circ \mathrm{d}^{k-1} = 0$ for each k.

$$\cdots \longrightarrow V^{k-1} \xrightarrow{\mathrm{d}^{k-1}} V^k \xrightarrow{\mathrm{d}^k} V^{k+1} \longrightarrow \cdots$$

This Hilbert complex is said to be *bounded* if d^k is a bounded linear map from W^k to W^{k+1} for each k, i.e., (W, d) is a cochain complex in the category of Hilbert spaces. It is said to be *closed* if the image $\mathrm{d}^k V^k$ is closed in W^{k+1} for each k.

Definition 2.2. Given two Hilbert complexes, (W, d) and (W', d'), a *morphism of Hilbert complexes* $f\colon W \to W'$ consists of a sequence of bounded linear maps $f^k\colon W^k \to W'^k$ such that $f^k V^k \subset V'^k$ and $\mathrm{d}'^k f^k = f^{k+1}\mathrm{d}^k$ for each k. That is, the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & V^k \arrow[r, "\mathrm{d}^k"] \arrow[d, "f^k"] & V^{k+1} \arrow[r] \arrow[d, "f^{k+1}"] & \cdots \\
\cdots \arrow[r] & V'^k \arrow[r, "\mathrm{d}'^k"] & V'^{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

By analogy with cochain complexes, it is possible to define notions of cocycles, coboundaries, harmonic forms, and cohomology spaces for Hilbert complexes.

Definition 2.3. Given a Hilbert complex (W, d), the space of *k-cocycles* is the kernel $\mathfrak{Z}^k = \ker \mathrm{d}^k$, the space of *$k$-coboundaries* is the image $\mathfrak{B}^k = \mathrm{d}^{k-1}V^{k-1}$, the *$k$th harmonic space* is the intersection $\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{B}^{k\perp_W}$, and the *$k$th reduced cohomology space* is the quotient $\mathfrak{Z}^k / \overline{\mathfrak{B}^k}$. When $\mathfrak{B}^k$ is closed, $\mathfrak{Z}^k/\mathfrak{B}^k$ is simply called the *kth cohomology space*, and is identical to reduced cohomology.

Remark 1. One can show that the harmonic space $\mathfrak{H}^k$ is isomorphic to the reduced cohomology space $\mathfrak{Z}^k/\overline{\mathfrak{B}^k}$. For a closed complex, this is identical to the usual cohomology space $\mathfrak{Z}^k/\mathfrak{B}^k$, since $\mathfrak{B}^k$ is closed for each k.

Definition 2.4. Given a morphism of Hilbert complexes $f\colon W \to W'$, the *induced map on (reduced) cohomology* is defined by $[z] \mapsto [fz]$, where $[z]$ denotes the (reduced) cohomology class of the cocycle z.

In general, the differentials d^k of a Hilbert complex may be unbounded linear maps. However, given an arbitrary Hilbert complex (W, d), it is always possible to construct a bounded complex having the same domains and maps, as follows.

Definition 2.5. Given a Hilbert complex (W, d), the *domain complex* (V, d) consists of the domains $V^k \subset W^k$, endowed with the graph inner product

$$\langle u, v \rangle_{V^k} = \langle u, v \rangle_{W^k} + \langle \mathrm{d}^k u, \mathrm{d}^k v \rangle_{W^{k+1}}.$$

Remark 2. Since d^k is a closed map, each V^k is closed with respect to the norm induced by the graph inner product. Also, each map d^k is bounded, since

$$\lVert \mathrm{d}^k v \rVert_{V^{k+1}} = \lVert \mathrm{d}^k v \rVert_{W^{k+1}} \leq \lVert v \rVert_{W^k} + \lVert \mathrm{d}^k v \rVert_{W^{k+1}} = \lVert v \rVert_{V^k}.$$

Thus, the domain complex is a bounded Hilbert complex; moreover, it is a closed complex if and only if (W, d) is closed.

For the remainder of the paper, we will follow the simplified notation used by Arnold, Falk, and Winther [3]: the W-inner product and norm will be written simply as $\langle \cdot, \cdot \rangle$ and $\lVert \cdot \rVert$, without subscripts, while the V-inner product and norm will be written explicitly as $\langle \cdot, \cdot \rangle_V$ and $\lVert \cdot \rVert_V$.

2.2. **Hodge decomposition and Poincaré inequality.** The Helmholtz decomposition states that a rapidly-decaying vector field on $\mathbb{R}^3$ can be decomposed into curl-free and divergence-free components, i.e., the vector field can be written as the sum of the gradient of a scalar potential and the curl of a vector potential. For differential forms, this is generalized by the Hodge decomposition, which states that any differential form can be written as a sum of exact, coexact, and harmonic components. Here, we recall an even further generalization of the Hodge decomposition to arbitrary Hilbert complexes; this immediately gives rise to an abstract version of the Poincaré inequality, which will be crucial to much of the later analysis.

Following Brüning and Lesch [8], we can decompose each space W^k in terms of orthogonal subspaces,

$$W^k = \mathfrak{Z}^k \oplus \mathfrak{Z}^{k\perp_W} = \mathfrak{Z}^k \cap \left(\overline{\mathfrak{B}^k} \oplus \mathfrak{B}^{k\perp_W}\right) \oplus \mathfrak{Z}^{k\perp_W} = \overline{\mathfrak{B}^k} \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_W},$$

where the final expression is known as the *weak Hodge decomposition*. For the domain complex (V, d), the spaces $\mathfrak{Z}^k$, $\mathfrak{B}^k$, and $\mathfrak{H}^k$ are the same as for (W, d), and consequently we get the decomposition

$$V^k = \overline{\mathfrak{B}^k} \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_V},$$

where $\mathfrak{Z}^{k\perp_V} = \mathfrak{Z}^{k\perp_W} \cap V^k$. In particular, if (W, d) is a closed Hilbert complex, then the image $\mathfrak{B}^k$ is a closed subspace, so we have the *strong Hodge decomposition*

$$W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_W},$$

and likewise for the domain complex,

$$V^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_V}.$$

From here on, following the notation of Arnold, Falk, and Winther [3], we will simply write $\mathfrak{Z}^{k\perp}$ in place of $\mathfrak{Z}^{k\perp_V}$ when there can be no confusion.

Lemma 2.6 (Abstract Poincaré Inequality)**.** *If (V, d) is a bounded, closed Hilbert complex, then there exists a constant c_P such that*

$$\lVert v \rVert_V \leq c_P \lVert \mathrm{d}^k v \rVert_V, \quad \forall v \in \mathfrak{Z}^{k\perp}.$$

Proof. The map d^k is a bounded bijection from $\mathfrak{Z}^{k\perp}$ to $\mathfrak{B}^{k+1}$, which are both closed subspaces, so the result follows immediately by applying the bounded inverse theorem. ☐

Corollary 2.7. *If (V, d) is the domain complex of a closed Hilbert complex (W, d), then*

$$\lVert v \rVert_V \leq c_P \lVert \mathrm{d}^k v \rVert, \quad \forall v \in \mathfrak{Z}^{k\perp}.$$

We close this subsection by defining the dual complex of a Hilbert complex, and recalling how the Hodge decomposition can be interpreted in terms of this complex.

Definition 2.8. Given a Hilbert complex (W, d), the *dual complex* (W^*, d^*) consists of the spaces $W_k^* = W^k$, and adjoint operators $\mathrm{d}_k^* = \left(\mathrm{d}^{k-1}\right)^* : V_k^* \subset W_k^* \to V_{k-1}^* \subset W_{k-1}^*$.

$$\cdots \longleftarrow V_{k-1}^* \xleftarrow{\mathrm{d}_k^*} V_k^* \xleftarrow{\mathrm{d}_{k+1}^*} V_{k+1}^* \longleftarrow \cdots$$

Remark 3. Since the arrows in the dual complex point in the opposite direction, this is a Hilbert chain complex rather than a cochain complex. (The chain property $\mathrm{d}_k^* \circ \mathrm{d}_{k+1}^* = 0$ follows easily from the cochain property $\mathrm{d}^k \circ \mathrm{d}^{k-1} = 0$.) Accordingly, we can define the *k-cycles* $\mathfrak{Z}_k^* = \ker \mathrm{d}_k^* = \mathfrak{B}^{k\perp_W}$ and *k-boundaries* $\mathfrak{B}_k^* = \mathrm{d}_{k+1}^* V_k^*$. The kth harmonic space can then be rewritten as $\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{Z}_k^*$; we also have $\mathfrak{Z}^k = \mathfrak{B}_k^{*\perp_W}$, and thus $\mathfrak{Z}^{k\perp_W} = \overline{\mathfrak{B}_k^*}$. Therefore, the weak Hodge decomposition can be written as

$$W^k = \overline{\mathfrak{B}^k} \oplus \mathfrak{H}^k \oplus \overline{\mathfrak{B}_k^*},$$

and in particular, for a closed Hilbert complex, the strong Hodge decomposition now becomes

$$W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{B}^*_k.$$

2.3. **The abstract Hodge Laplacian and mixed variational problem.** To obtain a "mixed version" of the familiar Poisson equation $-\Delta u = f$ for scalar functions, we now follow Arnold, Falk, and Winther [3] in defining an abstract version of the Hodge Laplacian for Hilbert complexes. The *abstract Hodge Laplacian* is the operator $L = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$, which is an unbounded operator $W^k \to W^k$ with domain

$$D_L = \left\{ u \in V^k \cap V^*_k \;\middle|\; \mathrm{d}u \in V^*_{k+1},\ \mathrm{d}^* u \in V^{k-1} \right\}.$$

If $u \in D_L$ solves $Lu = f$, then it satisfies the variational principle

$$\langle \mathrm{d}u, \mathrm{d}v \rangle + \langle \mathrm{d}^* u, \mathrm{d}^* v \rangle = \langle f, v \rangle, \quad \forall v \in V^k \cap V^*_k.$$

However, as noted by Arnold, Falk, and Winther [3], there are some difficulties in using this variational principle for a finite element approximation. First, it may be difficult to construct finite elements for the space $V^k \cap V^*_k$. A second concern is the well-posedness of the problem. If we take any harmonic test function $v \in \mathfrak{H}^k$, then the left-hand side vanishes, so $\langle f, v \rangle = 0$; hence, a solution only exists if $f \perp \mathfrak{H}^k$. Furthermore, for any $q \in \mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{Z}^*_k$, we have $\mathrm{d}q = 0$ and $\mathrm{d}^* q = 0$; therefore, if u is a solution, then so is $u + q$.

To avoid these existence and uniqueness issues, one can define instead the following mixed variational problem: Find $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ satisfying

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, \mathrm{d}\tau \rangle &= 0, && \forall \tau \in V^{k-1}, \\ \langle \mathrm{d}\sigma, v \rangle + \langle \mathrm{d}u, \mathrm{d}v \rangle + \langle p, v \rangle &= \langle f, v \rangle, && \forall v \in V^k, \\ \langle u, q \rangle &= 0, && \forall q \in \mathfrak{H}^k. \end{aligned} \tag{3}$$

Here, the first equation implies that $\sigma = \mathrm{d}^* u$, which weakly enforces the condition $u \in V^k \cap V^*_k$. Next, the second equation incorporates the additional term $\langle p, v \rangle$, which allows for solutions to exist even when $f \not\perp \mathfrak{H}^k$. Finally, the third equation fixes the issue of non-uniqueness by requiring $u \perp \mathfrak{H}^k$. The following result establishes the well-posedness of the problem (3).

Theorem 2.9 (Arnold, Falk, and Winther [3], Theorem 3.1)**.** *Let (W, d) be a closed Hilbert complex with domain complex (V, d). The mixed formulation of the abstract Hodge Laplacian is well-posed. That is, for any $f \in W^k$, there exists a unique $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ satisfying (3). Moreover,*

$$\|\sigma\|_V + \|u\|_V + \|p\| \leq c \|f\|,$$

where c is a constant depending only on the Poincaré constant c_P in Lemma 2.6.

To prove this, one observes that (3) can be rewritten as a standard variational problem (1) on the space $V^{k-1} \times V^k \times \mathfrak{H}^k$, with the bilinear form

$$B(\sigma, u, p; \tau, v, q) = \langle \sigma, \tau \rangle - \langle u, \mathrm{d}\tau \rangle + \langle \mathrm{d}\sigma, v \rangle + \langle \mathrm{d}u, \mathrm{d}v \rangle + \langle p, v \rangle - \langle u, q \rangle$$

and functional $F(\tau, v, q) = \langle f, v \rangle$. The well-posedness then follows immediately from the following theorem, which establishes the inf-sup condition for the bilinear form B.

Theorem 2.10 (Arnold, Falk, and Winther [3], Theorem 3.2)**.** *Let* (W, d) *be a closed Hilbert complex with domain complex* (V, d)*. There exists a constant* $\gamma > 0$*, depending only on the constant* c_P *in the Poincaré inequality (Lemma 2.6), such that for any* $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$*, there exists* $(\tau, v, q) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ *with*

$$B(\sigma, u, p; \tau, v, q) \geq \gamma \left(\lVert \sigma \rVert_V + \lVert u \rVert_V + \lVert p \rVert \right) \left(\lVert \tau \rVert_V + \lVert v \rVert_V + \lVert q \rVert \right).$$

From the well-posedness result, it follows that there exists a bounded solution operator $K \colon W^k \to W^k$ defined by $Kf = u$.

2.4. **Approximation by a subcomplex.** In order to obtain approximate numerical solutions to the mixed variational problem (3), Arnold, Falk, and Winther [3] suppose that one is given a (finite-dimensional) subcomplex $V_h \subset V$ of the domain complex: that is, $V_h^k \subset V^k$ is a Hilbert subspace for each k, and the inclusion mapping $i_h \colon V_h \hookrightarrow V$ is a morphism of Hilbert complexes. By analogy with the Galerkin method, one can then consider the mixed variational problem on the subcomplex: Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ satisfying

$$\begin{aligned} \langle \sigma_h, \tau \rangle - \langle u_h, \mathrm{d}\tau \rangle &= 0, & \forall \tau &\in V_h^{k-1}, \\ \langle \mathrm{d}\sigma_h, v \rangle + \langle \mathrm{d}u_h, \mathrm{d}v \rangle + \langle p_h, v \rangle &= \langle f, v \rangle, & \forall v &\in V_h^k, \\ \langle u_h, q \rangle &= 0, & \forall q &\in \mathfrak{H}_h^k. \end{aligned} \tag{4}$$

For the error analysis of this method, one more crucial assumption must be made: that there exists some Hilbert complex "projection" $\pi_h \colon V \to V_h$. We put "projection" in quotes because this need not be the actual orthogonal projection i_h^* with respect to the inner product; indeed, that projection is not generally a morphism of Hilbert complexes, since it may not commute with the differentials. However, the map π_h is V-bounded, surjective, and idempotent. It follows, then, that although it does not satisfy the optimality property of the true projection, it does still satisfy a *quasi-optimality* property, since

$$\lVert u - \pi_h u \rVert_V = \inf_{v \in V_h} \lVert (I - \pi_h)(u - v) \rVert_V \leq \lVert I - \pi_h \rVert \inf_{v \in V_h} \lVert u - v \rVert_V,$$

where the first step follows from the idempotence of π_h, i.e., $(I - \pi_h) v = 0$ for all $v \in V_h$. With this framework in place, the following error estimate can be established.

Theorem 2.11 (Arnold, Falk, and Winther [3], Theorem 3.9)**.** *Let* (V_h, d) *be a family of subcomplexes of the domain complex* (V, d) *of a closed Hilbert complex, parametrized by* h *and admitting uniformly* V*-bounded cochain projections, and let* $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ *be the solution of* (3) *and* $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ *the solution of problem* (4)*. Then*

$$\begin{aligned} &\lVert \sigma - \sigma_h \rVert_V + \lVert u - u_h \rVert_V + \lVert p - p_h \rVert \\ &\leq C \Big(\inf_{\tau \in V_h^{k-1}} \lVert \sigma - \tau \rVert_V + \inf_{v \in V_h^k} \lVert u - v \rVert_V + \inf_{q \in V_h^k} \lVert p - q \rVert_V + \mu \inf_{v \in V_h^k} \lVert P_{\mathfrak{B}} u - v \rVert_V \Big), \end{aligned}$$

where $\mu = \mu_h^k = \sup_{\substack{r \in \mathfrak{H}^k \\ \lVert r \rVert = 1}} \left\lVert \left(I - \pi_h^k \right) r \right\rVert$.

Therefore, if V_h is pointwise approximating, in the sense that $\inf_{v \in V_h} \lVert u - v \rVert \to 0$ as $h \to 0$ for every $u \in V$, then the numerical solution converges to the exact solution.

3. Analysis of variational crimes

In this section, we extend the results of Arnold, Falk, and Winther [3], summarized in the previous section, by removing the requirement for V_h to be a subcomplex of V. The key point of departure is in the map $i_h \colon V_h \hookrightarrow V$; rather than being an inclusion, we require only that it is an injective morphism of Hilbert complexes, with the property that $\pi_h \circ i_h$ is the identity. (The latter requirement simply corresponds to the earlier condition that π_h be idempotent in the case of subcomplexes.) After stating some basic results about complexes equipped with such maps, we develop error estimates for the mixed variational problem and eigenvalue problem on V_h. These estimates contain two additional error terms, in addition to those in the analysis of Arnold, Falk, and Winther [3]. These extra terms, analogous to those in the Strang lemmas for generalized Galerkin methods, measure the "severity" of two variational crimes: first, how well the right-hand side $i_h f_h$ approximates f; and second, the extent to which i_h fails to be unitary.

3.1. Approximation by an arbitrary complex.

In order to approximate a Hilbert complex (W, d), suppose we have another Hilbert complex (W_h, d_h), along with a pair of morphisms: an injection $i_h \colon W_h \hookrightarrow W$ and a projection $\pi_h \colon W \to W_h$, such that $\pi_h^k \circ i_h^k$ is the identity on W_h^k for each k. Recall that, by Definition 2.2 of a Hilbert complex morphism, the maps i_h^k and π_h^k must be bounded for each k. The relationships among the domains and maps are illustrated in the following diagram:

$$\begin{tikzcd} \cdots \arrow[r] & V^k \arrow[r, "\mathrm{d}^k"] \arrow[d, shift left, "\pi_h^k"] & V^{k+1} \arrow[r] \arrow[d, shift left, "\pi_h^{k+1}"] & \cdots \\ \cdots \arrow[r] & V_h^k \arrow[r, "\mathrm{d}_h^k"] \arrow[u, shift left, "i_h^k"] & V_h^{k+1} \arrow[r] \arrow[u, shift left, "i_h^{k+1}"] & \cdots . \end{tikzcd}$$

Arnold, Falk, and Winther [3] consider the case where $W_h \subset W$ is a subcomplex, and i_h is the inclusion of W_h into W. In this special case, i_h is unitary (i.e., an isometry), since for all $u, v \in W_h^k$, we have $\langle i_h u, i_h v \rangle = \langle u, v \rangle = \langle u, v \rangle_h$. Indeed, if i_h is unitary, then we can simply identify W_h with the subcomplex $i_h W_h \subset W$. More generally, though, we will consider cases where $W_h \not\subset W$, and where i_h is not necessarily unitary.

We begin by demonstrating some basic facts about these approximations.

Theorem 3.1. *If (W, d) is a bounded Hilbert complex, then so is (W_h, d_h).*

Proof. $\lVert \mathrm{d}_h^k \rVert = \lVert \pi_h^{k+1} i_h^{k+1} \mathrm{d}_h^k \rVert = \lVert \pi_h^{k+1} \mathrm{d}^k i_h^k \rVert \leq \lVert \pi_h^{k+1} \rVert \lVert \mathrm{d}^k \rVert \lVert i_h^k \rVert < \infty$. □

Theorem 3.2. *If (W, d) is a closed Hilbert complex, then so is (W_h, d_h).*

Proof. Assume that (W, d) is closed, so that each coboundary space $\mathfrak{B}^k$ is closed in W^k. Now, since i_h is a morphism, if $v_h \in \mathfrak{B}_h^k$ then $i_h v_h \in \mathfrak{B}^k$, so $\mathfrak{B}_h^k \subset i_h^{-1} \mathfrak{B}^k$. Conversely, since π_h is a morphism, if $i_h v_h \in \mathfrak{B}^k$ then $v_h = \pi_h i_h v_h \in \mathfrak{B}_h^k$, so $i_h^{-1} \mathfrak{B}^k \subset \mathfrak{B}_h^k$. Therefore, $\mathfrak{B}_h^k = i_h^{-1} \mathfrak{B}^k$, and since i_h is bounded (and hence continuous), it follows that $\mathfrak{B}_h^k$ is closed. □

Since $\pi_h^k \circ i_h^k = \mathrm{id}_{W_h^k}$, this composition induces the identity map on the reduced cohomology space $\mathfrak{Z}_h^k / \overline{\mathfrak{B}_h^k}$; thus i_h induces an injection on reduced cohomology, while π_h induces a surjection. We now show that, given a certain approximation condition on the harmonic spaces $\mathfrak{H}^k$, these induced maps are in fact isomorphisms (which are inverses of one another, since their composition is the identity).

Theorem 3.3. *Let* (W, d) *and* (W_h, d_h) *be Hilbert complexes, with morphisms* $i_h \colon W_h \hookrightarrow W$ *and* $\pi_h \colon W \to W_h$ *such that* $\pi_h^k \circ i_h^k = \mathrm{id}_{W_h^k}$ *for each* k*. If, for all* k*,*

$$\left\| q - i_h^k \pi_h^k q \right\| < \left\| q \right\|, \quad \forall q \in \mathfrak{H}^k, \ q \neq 0,$$

then π_h *(and thus* i_h*) induces an isomorphism on the reduced cohomology spaces.*

Proof. Since π_h induces a surjection on reduced cohomology, it suffices to show that this is also an injection. That is, given $z \in \mathfrak{Z}^k$ with $\pi_h z \in \overline{\mathfrak{B}_h^k}$, we must demonstrate that $z \in \overline{\mathfrak{B}^k}$. Using the weak Hodge decomposition, write $z = q + b$, where $q \in \mathfrak{H}^k$ and $b \in \overline{\mathfrak{B}^k}$. By assumption, $\pi_h z \in \overline{\mathfrak{B}_h^k}$, and since π_h is a morphism, $\pi_h b \in \overline{\mathfrak{B}_h^k}$ as well. Thus, $\pi_h q = \pi_h z - \pi_h b \in \overline{\mathfrak{B}_h^k}$, and since i_h is also a morphism, $i_h \pi_h q \in \overline{\mathfrak{B}^k} \perp \mathfrak{H}^k$. Therefore, $i_h \pi_h q \perp q$, which implies that q violates the inequality above, so we must have $q = 0$ and hence $z \in \overline{\mathfrak{B}^k}$. □

Corollary 3.4. *If* (W, d) *and* (W_h, d_h) *are closed Hilbert complexes, with morphisms* π_h *and* i_h *satisfying the above assumptions, then* π_h *(and thus* i_h*) induces an isomorphism on cohomology.*

Remark 4. This result is slightly more general than Arnold, Falk, and Winther [3, Theorem 3.4], which only treated the case of a bounded, closed Hilbert complex. However, the proof is essentially identical.

Next, suppose that (V, d) and (V_h, d_h) are bounded, closed Hilbert complexes; for example, they may be the domain complexes corresponding, respectively, to closed complexes (W, d) and (W_h, d_h). We now show that the Poincaré inequality for (V_h, d_h) can be written entirely in terms of the Poincaré constant for (V, d), denoted by c_P, along with the operator norms of i_h and π_h.

Theorem 3.5. *Let* (V, d) *and* (V_h, d_h) *be bounded, closed Hilbert complexes, with morphisms* $i_h \colon V_h \hookrightarrow V$ *and* $\pi_h \colon V \to V_h$ *such that* $\pi_h^k \circ i_h^k = \mathrm{id}_{V_h^k}$ *for each* k*. Then*

$$\left\| v_h \right\|_{V_h} \leq c_P \left\| \pi_h^k \right\| \left\| i_h^{k+1} \right\| \left\| \mathrm{d}_h v_h \right\|_{V_h}, \quad \forall v_h \in \mathfrak{Z}_h^{k\perp}.$$

Proof. Given $v_h \in \mathfrak{Z}_h^{k\perp}$, let $z \in \mathfrak{Z}^{k\perp}$ be the unique element such that $\mathrm{d}z = \mathrm{d} i_h v_h = i_h \mathrm{d}_h v_h$. Then, applying the abstract Poincaré inequality on V,

$$\left\| z \right\|_V \leq c_P \left\| \mathrm{d}z \right\|_V = c_P \left\| i_h \mathrm{d}_h v_h \right\|_V \leq c_P \left\| i_h^{k+1} \right\| \left\| \mathrm{d}_h v_h \right\|_{V_h}.$$

It now suffices to show $\left\| v_h \right\|_{V_h} \leq \left\| \pi_h^k \right\| \left\| z \right\|_V$. Observe that $v_h - \pi_h z \in V_h^k$, and furthermore,

$$\mathrm{d}_h \pi_h z = \pi_h \mathrm{d}z = \pi_h i_h \mathrm{d}_h v_h = \mathrm{d}_h v_h,$$

so $v_h - \pi_h z \in \mathfrak{Z}_h^k \perp v_h$. Therefore,

$$\left\| v_h \right\|_{V_h}^2 = \left\langle v_h, \pi_h z \right\rangle_{V_h} + \left\langle v_h, v_h - \pi_h z \right\rangle_{V_h} = \left\langle v_h, \pi_h z \right\rangle_{V_h} \leq \left\| v_h \right\|_{V_h} \left\| \pi_h^k \right\| \left\| z \right\|_V,$$

and the result follows. □

Corollary 3.6. *If* (V, d) *and* (V_h, d_h) *are the domain complexes corresponding, respectively, to closed Hilbert complexes* (W, d) *and* (W_h, d_h)*, then*

$$\left\| v_h \right\|_{V_h} \leq c_P \left\| \pi_h^k \right\| \left\| i_h^{k+1} \right\| \left\| \mathrm{d}_h v_h \right\|_h, \quad \forall v_h \in \mathfrak{Z}_h^{k\perp}.$$

Finally, given the importance of the projection morphism π_h in finite element exterior calculus, we now prove a short but useful result on the existence of such projections. In particular, the next theorem states how a projection morphism on another complex, W', can be "pulled back" to obtain one on W, as pictured in the following diagram:

$$\begin{tikzcd}
W \arrow[rr, "f"] \arrow[dr, dashed, shift left, "\pi_h"] & & W' \arrow[dl, shift right, "\pi'_h"'] \\
& W_h \arrow[ul, shift left, "i_h"] \arrow[ur, shift right, "i'_h"'] &
\end{tikzcd}.$$

In Section 4, this will allow us to obtain a projection morphism for the de Rham complex on a manifold, by pulling back the usual projection defined on its piecewise-linear triangulation.

Theorem 3.7. *Let* (W, d) *and* (W_h, d_h) *be Hilbert complexes with an injection morphism* $i_h \colon W_h \hookrightarrow W$. *Suppose there exists another complex* (W', d') *and a morphism* $f \colon W \to W'$, *such that* $i'_h = f \circ i_h \colon W_h \hookrightarrow W'$ *is injective and has a corresponding projection morphism* $\pi'_h \colon W' \to W_h$ *with* $\pi'_h \circ i'_h = \mathrm{id}_{W_h}$. *Then there also exists a projection morphism* $\pi_h \colon W \to W_h$ *such that* $\pi_h \circ i_h = \mathrm{id}_{W_h}$.

Proof. Take $\pi_h = \pi'_h \circ f$. Then $\pi_h \circ i_h = \pi'_h \circ f \circ i_h = \pi'_h \circ i'_h = \mathrm{id}_{W_h}$. □

3.2. **Modified inner product and Hodge decomposition.** As noted in the previous section, this generalized framework introduces some new complications, due to the possible non-unitarity of i_h. The following result shows that the subspace $i_h W_h \subset W$ can be identified with W_h, endowed with a *modified* inner product $\langle J_h \cdot, \cdot \rangle_h$ instead of $\langle \cdot, \cdot \rangle_h$. This defines a modified Hilbert complex, which will be denoted by $(i_h^* W, \mathrm{d}_h)$.

Theorem 3.8. *Let* $i_h \colon W_h \hookrightarrow W$ *be a morphism of Hilbert complexes, and define* $J_h^k = i_h^{k*} i_h^k \colon W_h^k \to W_h^k$ *for each* k. *Then*

$$\langle J_h u_h, v_h \rangle_h = \langle i_h u_h, i_h v_h \rangle, \quad \forall u_h, v_h \in W_h^k,$$

is an inner product, which defines a Hilbert space structure on W_h^k.

Proof. $\langle i_h u_h, i_h v_h \rangle = \langle i_h^* i_h u_h, v_h \rangle_h = \langle J_h u_h, v_h \rangle_h$. This is an inner product, since i_h is linear and injective. Moreover, W_h^k is closed with respect to the induced norm, since $\lVert i_h v_h \rVert \leq \lVert i_h \rVert \, \lVert v_h \rVert$ and i_h is bounded, so this is indeed a Hilbert space. □

Remark 5. We use the notation J_h due to the similarity with the Jacobian determinant used in the "change of variables" formula for integration. Note that, although each $J_h^k \colon W_h^k \to W_h^k$ is a bounded linear map, J_h is not necessarily a Hilbert complex automorphism. This is because, in general, d commutes with i_h but not with its adjoint i_h^*. Also, clearly i_h^k is unitary if and only if $J_h^k = \mathrm{id}_{W_h^k}$.

Now, if i_h does not preserve the inner product, in particular it does not preserve orthogonality: that is, $u_h \perp v_h$ does not imply $i_h u_h \perp i_h v_h$. This has significant implications for the Hodge decomposition, since although $W_h^k = \overline{\mathfrak{B}_h^k} \oplus \mathfrak{H}_h^k \oplus \mathfrak{Z}_h^{k\perp_{W_h}}$ is W_h-orthogonal, it is generally not $i_h^* W$-orthogonal. Therefore, we define the new, modified subspaces

$$\mathfrak{H}_h'^k = \left\{ z \in \mathfrak{Z}_h^k \;\middle|\; i_h z \perp i_h \mathfrak{B}_h^k \right\}, \qquad \mathfrak{Z}_h^{k\perp' W} = \left\{ v \in W_h^k \;\middle|\; i_h v \perp i_h \mathfrak{Z}_h^k \right\}.$$

This gives a modified Hodge decomposition $W_h^k = \overline{\mathfrak{B}_h^k} \oplus \mathfrak{H}_h'^k \oplus \mathfrak{Z}_h^{k\perp' W}$, which is no longer necessarily W_h-orthogonal, but is now $i_h^* W$-orthogonal. As before, this also gives a modified Hodge decomposition for the domain complex $V_h^k = \overline{\mathfrak{B}_h^k} \oplus \mathfrak{H}_h'^k \oplus \mathfrak{Z}_h^{k\perp'}$.

3.3. **Stability and convergence of the mixed method.** Let (W, d) be a closed Hilbert complex with domain complex (V, d). To approximate a solution to the mixed variational problem (3), suppose that (W_h, d_h) is another Hilbert complex with domain complex (V_h, d_h), and that we have morphisms $i_h \colon V_h \hookrightarrow V$ and $\pi_h \colon V \to V_h$ such that $\pi_h^k \circ i_h^k = \mathrm{id}_{V_h^k}$ for each k. We assume that i_h is W-bounded, so that it also can be extended to $W_h \hookrightarrow W$, but that π_h might only be V-bounded. Then consider the solution of the following mixed variational problem: Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ satisfying

$$
\begin{aligned}
\langle \sigma_h, \tau_h \rangle_h - \langle u_h, \mathrm{d}_h \tau_h \rangle_h &= 0, & \forall \tau_h &\in V_h^{k-1},\\
\langle \mathrm{d}_h \sigma_h, v_h \rangle_h + \langle \mathrm{d}_h u_h, \mathrm{d}_h v_h \rangle_h + \langle p_h, v_h \rangle_h &= \langle f_h, v_h \rangle_h, & \forall v_h &\in V_h^k,\\
\langle u_h, q_h \rangle_h &= 0, & \forall q_h &\in \mathfrak{H}_h^k.
\end{aligned} \tag{5}
$$

This corresponds to the generalized variational problem (2) with bilinear form

$$
\begin{aligned}
B_h(\sigma_h, u_h, p_h; \tau_h, v_h, q_h) &= \langle \sigma_h, \tau_h \rangle_h - \langle u_h, \mathrm{d}_h \tau_h \rangle_h \\
&\quad + \langle \mathrm{d}_h \sigma_h, v_h \rangle_h + \langle \mathrm{d}_h u_h, \mathrm{d}_h v_h \rangle_h + \langle p_h, v_h \rangle_h - \langle u_h, q_h \rangle_h
\end{aligned}
$$

and functional $F_h(\tau_h, v_h, q_h) = \langle f_h, v_h \rangle_h$. The following theorem establishes the inf-sup condition for the mixed method (5).

Theorem 3.9. *Let (V, d) be the domain complex of a closed Hilbert complex (W, d), and let (V_h, d_h) be a family of domain complexes of closed Hilbert complexes (W_h, d_h), equipped with uniformly W-bounded inclusion morphisms $i_h \colon V_h \hookrightarrow V$ and V-bounded projection morphisms $\pi_h \colon V \to V_h$ satisfying $\pi_h^k \circ i_h^k = \mathrm{id}_{V_h^k}$. Then there exists a constant $\gamma_h > 0$, depending only on c_P and the norms of i_h and π_h, such that for any $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$, there exists $(\tau_h, v_h, q_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ where*

$$
\begin{aligned}
&B_h(\sigma_h, u_h, p_h; \tau_h, v_h, q_h) \\
&\qquad \geq \gamma_h \left(\lVert \sigma_h \rVert_{V_h} + \lVert u_h \rVert_{V_h} + \lVert p_h \rVert_h \right) \left(\lVert \tau_h \rVert_{V_h} + \lVert v_h \rVert_{V_h} + \lVert q_h \rVert_h \right).
\end{aligned}
$$

Proof. This is just Theorem 2.10 applied to the Hilbert complex (V_h, d_h), combined with the fact that the Poincaré constant is $c_P \lVert \pi_h \rVert \lVert i_h \rVert$ by Theorem 3.5. ☐

Remark 6. Since we have assumed that the morphisms i_h and π_h are uniformly bounded with respect to h, it follows that the inf-sup constants γ_h can be bounded below by some constant, which is independent of h.

The goal, for the remainder of this section, will be to control the error

$$
\lVert \sigma - i_h \sigma_h \rVert_V + \lVert u - i_h u_h \rVert_V + \lVert p - i_h p_h \rVert,
$$

where (σ, u, p) is a solution to (3) and (σ_h, u_h, p_h) is a solution to (5). To do this, it will be helpful to introduce the following modified mixed problem on $i_h^* V$: Find $(\sigma_h', u_h', p_h') \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h'^k$ satisfying

$$
\begin{aligned}
\langle J_h \sigma_h', \tau_h \rangle_h - \langle J_h u_h', \mathrm{d}_h \tau_h \rangle_h &= 0, & \forall \tau_h &\in V_h^{k-1},\\
\langle J_h \mathrm{d}_h \sigma_h', v_h \rangle_h + \langle J_h \mathrm{d}_h u_h', \mathrm{d}_h v_h \rangle_h + \langle J_h p_h', v_h \rangle_h &= \langle i_h^* f, v_h \rangle_h, & \forall v_h &\in V_h^k,\\
\langle J_h u_h', q_h' \rangle_h &= 0, & \forall q_h' &\in \mathfrak{H}_h'^k.
\end{aligned} \tag{6}
$$

This has the corresponding bilinear form

$$B_h'\left(\sigma_h', u_h', p_h'; \tau_h, v_h, q_h'\right) = \left\langle J_h \sigma_h', \tau_h \right\rangle_h - \left\langle J_h u_h', \mathrm{d}_h \tau_h \right\rangle_h \\ + \left\langle J_h \mathrm{d}_h \sigma_h', v_h \right\rangle_h + \left\langle J_h \mathrm{d}_h u_h', \mathrm{d}_h v_h \right\rangle_h + \left\langle J_h p_h', v_h \right\rangle_h - \left\langle J_h u_h', q_h' \right\rangle_h,$$

and the functional $F_h'\left(\tau_h, v_h, q_h'\right) = \left\langle i_h^* f, v_h \right\rangle_h$.

This is precisely equivalent to the mixed problem on the subcomplex $i_h V_h \subset V$, which has the bounded cochain projection $i_h \circ \pi_h \colon V \to i_h V_h$. Therefore, the stability and convergence analysis of Arnold, Falk, and Winther [3] can be applied immediately to this modified discrete problem. In the end, we will obtain the desired bound by applying the triangle inequality,

$$\begin{aligned} (7)\quad & \left\| \sigma - i_h \sigma_h \right\|_V + \left\| u - i_h u_h \right\|_V + \left\| p - i_h p_h \right\| \\ &\leq \left\| \sigma - i_h \sigma_h' \right\|_V + \left\| u - i_h u_h' \right\|_V + \left\| p - i_h p_h' \right\| \\ &\quad + \left\| i_h \left(\sigma_h - \sigma_h' \right) \right\|_V + \left\| i_h \left(u_h - u_h' \right) \right\|_V + \left\| i_h \left(p_h - p_h' \right) \right\|. \end{aligned}$$

Observe that, since i_h is bounded, we can write

$$\left\| i_h \left(\sigma_h - \sigma_h' \right) \right\|_V + \left\| i_h \left(u_h - u_h' \right) \right\|_V + \left\| i_h \left(p_h - p_h' \right) \right\| \\ \leq C \left(\left\| \sigma_h - \sigma_h' \right\|_{V_h} + \left\| u_h - u_h' \right\|_{V_h} + \left\| p_h - p_h' \right\|_h \right),$$

so it will suffice to control the error between solutions to (5) and (6) in V_h.

Theorem 3.10. *Under the assumptions of Theorem 3.9, suppose that* $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ *is a solution to* (5) *and* $(\sigma_h', u_h', p_h') \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h'^k$ *is a solution to* (6). *Then*

$$\left\| \sigma_h - \sigma_h' \right\|_{V_h} + \left\| u_h - u_h' \right\|_{V_h} + \left\| p_h - p_h' \right\|_h \leq C \left(\left\| f_h - i_h^* f \right\|_h + \left\| I - J_h \right\| \left\| f \right\| \right).$$

Proof. For any $(\tau, v, q) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$, we can write

$$B_h\left(\sigma_h - \tau, u_h - v, p_h - q; \tau_h, v_h, q_h\right) = B_h\left(\sigma_h - \sigma_h', u_h - u_h', p_h - p_h'; \tau_h, v_h, q_h\right) \\ + B_h\left(\sigma_h' - \tau, u_h' - v, p_h' - q; \tau_h, v_h, q_h\right).$$

Ignoring the first term momentarily, observe for the second term that

$$B_h\left(\sigma_h', u_h', p_h'; \tau_h, v_h, q_h\right) = B_h'\left(\sigma_h', u_h', p_h'; \tau_h, v_h, q_h\right) \\ + \left\langle \left(I - J_h\right) \sigma_h', \tau_h \right\rangle_h - \left\langle \left(I - J_h\right) u_h', \mathrm{d}_h \tau_h \right\rangle_h + \left\langle \left(I - J_h\right) \mathrm{d}_h \sigma_h', v_h \right\rangle_h \\ + \left\langle \left(I - J_h\right) \mathrm{d}_h u_h', \mathrm{d}_h v_h \right\rangle_h + \left\langle \left(I - J_h\right) p_h', v_h \right\rangle_h - \left\langle \left(I - J_h\right) u_h', q_h \right\rangle_h,$$

so by the variational principles (5) and (6),

$$\begin{aligned} B_h'\left(\sigma_h', u_h', p_h'; \tau_h, v_h, q_h\right) &= \left\langle i_h^* f, v_h \right\rangle_h - \left\langle J_h u_h', q_h \right\rangle_h, \\ B_h\left(\sigma_h, u_h, p_h; \tau_h, v_h, q_h\right) &= \left\langle f_h, v_h \right\rangle_h. \end{aligned}$$

Therefore,

$$B_h\left(\sigma_h - \sigma_h', u_h - u_h', p_h - p_h'; \tau_h, v_h, q_h\right) = \left\langle f_h - i_h^* f, v_h \right\rangle_h + \left\langle u_h', q_h \right\rangle_h \\ - \left\langle \left(I - J_h\right) \sigma_h', \tau_h \right\rangle_h + \left\langle \left(I - J_h\right) u_h', \mathrm{d}_h \tau_h \right\rangle_h - \left\langle \left(I - J_h\right) \mathrm{d}_h \sigma_h', v_h \right\rangle_h \\ - \left\langle \left(I - J_h\right) \mathrm{d}_h u_h', \mathrm{d}_h v_h \right\rangle_h - \left\langle \left(I - J_h\right) p_h', v_h \right\rangle_h,$$

so using the boundedness of the bilinear form and Cauchy–Schwarz, we get the upper bound

$$\begin{aligned}
&B_h\left(\sigma_h-\tau, u_h-v, p_h-q; \tau_h, v_h, q_h\right)\\
&\quad\le C\Big(\left\|f_h-i_h^* f\right\|_h+\left\|P_{\mathfrak{H}_h} u_h'\right\|_h+\left\|I-J_h\right\|\left(\left\|\sigma_h'\right\|_{V_h}+\left\|u_h'\right\|_{V_h}+\left\|p_h'\right\|_h\right)\\
&\qquad+\left\|\sigma_h'-\tau\right\|_{V_h}+\left\|u_h'-v\right\|_{V_h}+\left\|p_h'-q\right\|_h\Big)\Big(\left\|\tau_h\right\|_{V_h}+\left\|v_h\right\|_{V_h}+\left\|q_h\right\|_h\Big).
\end{aligned}$$

Next, Theorem 3.9 gives the lower bound

$$\begin{aligned}
&B_h\left(\sigma_h-\tau, u_h-v, p_h-q; \tau_h, v_h, q_h\right)\\
&\quad\ge \gamma_h\left(\left\|\sigma_h-\tau\right\|_{V_h}+\left\|u_h-v\right\|_{V_h}+\left\|p_h-q\right\|_h\right)\left(\left\|\tau_h\right\|_{V_h}+\left\|v_h\right\|_{V_h}+\left\|q_h\right\|_h\right)
\end{aligned}$$

for some $(\tau_h, v_h, q_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$, where γ_h can be bounded below independently of h. Therefore, combining the upper and lower bounds and dividing out $\left\|\tau_h\right\|_{V_h}+\left\|v_h\right\|_{V_h}+\left\|q_h\right\|_h$, we get

$$\begin{aligned}
&\left\|\sigma_h-\tau\right\|_{V_h}+\left\|u_h-v\right\|_{V_h}+\left\|p_h-q\right\|_h\\
&\quad\le C\Big(\left\|f_h-i_h^* f\right\|_h+\left\|P_{\mathfrak{H}_h} u_h'\right\|_h+\left\|I-J_h\right\|\left(\left\|\sigma_h'\right\|_{V_h}+\left\|u_h'\right\|_{V_h}+\left\|p_h'\right\|_h\right)\\
&\qquad\qquad+\left\|\sigma_h'-\tau\right\|_{V_h}+\left\|u_h'-v\right\|_{V_h}+\left\|p_h'-q\right\|_h\Big).
\end{aligned}$$

This expression can be simplified considerably by choosing $\tau=\sigma_h'$, $v=u_h'$, and $q=P_{\mathfrak{H}_h} p_h'$, so applying the triangle inequality gives the error estimate

$$\begin{aligned}
&\left\|\sigma_h-\sigma_h'\right\|_{V_h}+\left\|u_h-u_h'\right\|_{V_h}+\left\|p_h-p_h'\right\|_h\\
&\qquad\le C\left(\left\|f_h-i_h^* f\right\|_h+\left\|P_{\mathfrak{H}_h} u_h'\right\|_h+\left\|I-J_h\right\|\|f\|+\left\|p_h'-q\right\|_h\right).
\end{aligned}$$

All that remains is to deal with the terms $\left\|P_{\mathfrak{H}_h} u_h'\right\|_h$ and $\left\|p_h'-q\right\|_h$. First, since u_h' is $i_h^* V$-orthogonal to $\mathfrak{H}_h'^k$, the modified Hodge decomposition lets us write $u_h'=u_{\mathfrak{B}}'+u_{\perp}'$, where $u_{\mathfrak{B}}' \in \mathfrak{B}_h^k$ and $u_{\perp}' \in \mathfrak{Z}_h^{k\perp\prime}$. Now, observe that $P_{\mathfrak{H}_h} u_{\mathfrak{B}}'=0$ since $\mathfrak{B}_h^k \perp \mathfrak{H}_h^k$, and furthermore $P_{\mathfrak{H}_h} J_h u_{\perp}'=0$ since $u_{\perp}' \in \mathfrak{Z}_h^{k\perp\prime}$ implies $J_h u_{\perp}' \perp \mathfrak{Z}_h^k$. Therefore,

$$\left\|P_{\mathfrak{H}_h} u_h'\right\|_h=\left\|P_{\mathfrak{H}_h} u_{\perp}'\right\|_h=\left\|P_{\mathfrak{H}_h}\left(I-J_h\right) u_{\perp}'\right\|_h \le C\left\|I-J_h\right\|\|f\| .$$

Next, since $p_h' \in \mathfrak{H}_h'^k \subset \mathfrak{Z}_h^k$, the Hodge decomposition gives $p_h'=P_{\mathfrak{B}_h} p_h'+P_{\mathfrak{H}_h} p_h'=P_{\mathfrak{B}_h} p_h'+q$. Also, similar to the previous term, since $p_h' \in \mathfrak{H}_h'^k$ we have $J_h p_h' \perp \mathfrak{B}_h^k$, so $P_{\mathfrak{B}_h} J_h p_h'=0$. Thus,

$$\left\|p_h'-q\right\|_h=\left\|P_{\mathfrak{B}_h} p_h'\right\|_h=\left\|P_{\mathfrak{B}_h}\left(I-J_h\right) p_h'\right\|_h \le C\left\|I-J_h\right\|\|f\| .$$

Therefore, these two terms can be combined with the existing $\left\|I-J_h\right\|\|f\|$ term, leaving the final error estimate,

$$\left\|\sigma_h-\sigma_h'\right\|_{V_h}+\left\|u_h-u_h'\right\|_{V_h}+\left\|p_h-p_h'\right\|_h \le C\left(\left\|f_h-i_h^* f\right\|_h+\left\|I-J_h\right\|\|f\|\right),$$

as desired, which completes the proof. ∎

Corollary 3.11. *If $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ is a solution to* (3) *and $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ is a solution to* (5)*, then*

$$\begin{aligned}
&\|\sigma - i_h\sigma_h\|_V + \|u - i_h u_h\|_V + \|p - i_h p_h\| \\
&\leq C\Big(\inf_{\tau \in i_h V_h^{k-1}} \|\sigma - \tau\|_V + \inf_{v \in i_h V_h^k} \|u - v\|_V + \inf_{q \in i_h V_h^k} \|p - q\|_V + \mu \inf_{v \in i_h V_h^k} \|P_{\mathfrak{B}} u - v\|_V \\
&\qquad + \|f_h - i_h^* f\|_h + \|I - J_h\| \, \|f\| \Big),
\end{aligned}$$

where $\mu = \mu_h^k = \sup_{\substack{r \in \mathfrak{H}^k \\ \|r\|=1}} \big\| \big(I - i_h^k \pi_h^k\big) r \big\|$.

Proof. Use the triangle inequality, as in (7), and then apply Theorem 2.11 and Theorem 3.10 to bound the respective error terms. □

This theorem establishes convergence, as long as our approximations satisfy $\|I - J_h\| \to 0$ and $\|f_h - i_h^* f\|_h \to 0$ when $h \to 0$. This raises the question of how to choose $f_h \in V_h^k$; although clearly $f_h = i_h^* f$ will work, in many cases this cannot be computed efficiently. The next result demonstrates that, if $\Pi_h \colon W^k \to W_h^k$ is any bounded linear projection (i.e., satisfying $\Pi_h \circ i_h^k = \mathrm{id}_{W_h^k}$), then simply choosing $f_h = \Pi_h f$ is sufficient to get a quasi-optimally convergent solution.

Theorem 3.12. *If $\Pi_h \colon W^k \to W_h^k$ is a family of linear projections, bounded uniformly with respect to h, then we have the inequality*

$$\|\Pi_h f - i_h^* f\|_h \leq C\Big(\|I - J_h\| \, \|f\| + \inf_{\phi \in i_h W_h^k} \|f - \phi\| \Big).$$

Proof. Using the triangle inequality, we write

$$\begin{aligned}
\|(\Pi_h - i_h^*) f\|_h &\leq \|(\Pi_h - i_h^* i_h \Pi_h) f\|_h + \|(i_h^* - i_h^* i_h \Pi_h) f\|_h \\
&= \|(I - i_h^* i_h) \Pi_h f\|_h + \|i_h^* (I - i_h \Pi_h) f\|_h \\
&\leq \|I - J_h\| \, \|\Pi_h f\|_h + \|i_h^*\| \, \|(I - i_h \Pi_h) f\| \\
&\leq C\Big(\|I - J_h\| \, \|f\| + \inf_{\phi \in i_h W_h^k} \|f - \phi\| \Big),
\end{aligned}$$

where the final step follows from the W-boundedness of Π_h and the quasi-optimality property of $I - i_h \Pi_h$, i.e., $(I - i_h \Pi_h) f = (I - i_h \Pi_h)(f - \phi)$ for any $\phi \in i_h W_h^k$. □

3.4. **Remarks on obtaining improved error estimates.** Arnold, Falk, and Winther [3] were also able to obtain improved error estimates by making some additional assumptions: namely, that π_h is W-bounded rather than merely V-bounded, and that the Hilbert complex V satisfies a certain compactness property. With these assumptions, the continuous solution operator $K \colon W^k \to W^k$ becomes a compact operator, and hence converts the pointwise convergence of $I - \pi_h \to 0$ (which follows from the quasi-optimality property) to norm convergence. This norm convergence is essential for applying the so-called "Aubin–Nitsche trick" (also known as "L^2 lifting"), where one obtains improved estimates by applying the solution operator to the error term itself. Roughly speaking, one needs norm convergence, rather than pointwise convergence, since the solution operator is being applied to quantities that depend on the parameter h.

However, there are no such improved estimates for the additional error terms obtained in the previous subsection. Essentially, this is because norm convergence is already required for $\|I - J_h\| \to 0$ as $h \to 0$, and there is no analogous quasi-optimality result for J_h as there is for π_h. Therefore, these terms remain the same, and the improved estimates only apply to the terms already analyzed by Arnold, Falk, and Winther [3] for the subcomplex case.

3.5. **Convergence of the eigenvalue problem.** While we have primarily focused on the numerical approximation of the mixed variational problem, Arnold, Falk, and Winther [3] also analyzed an eigenvalue problem associated to the Hodge Laplacian. The extension of their eigenvalue convergence result to non-subcomplexes is fairly straightforward, and follows from the results already given in this section, as we will now show.

Consider the eigenvalue problem

$$
\begin{aligned}
\langle \sigma, \tau \rangle - \langle u, \mathrm{d}\tau \rangle &= 0, && \forall \tau \in V^{k-1}, \\
\langle \mathrm{d}\sigma, v \rangle + \langle \mathrm{d}u, \mathrm{d}v \rangle + \langle p, v \rangle &= \lambda \langle u, v \rangle, && \forall v \in V^k, \\
\langle u, q \rangle &= 0, && \forall q \in \mathfrak{H}^k,
\end{aligned}
\tag{8}
$$

the discrete problem

$$
\begin{aligned}
\langle \sigma_h, \tau_h \rangle_h - \langle u_h, \mathrm{d}_h\tau_h \rangle_h &= 0, && \forall \tau_h \in V_h^{k-1}, \\
\langle \mathrm{d}_h\sigma_h, v_h \rangle_h + \langle \mathrm{d}_h u_h, \mathrm{d}_h v_h \rangle_h + \langle p_h, v_h \rangle_h &= \lambda_h \langle u_h, v_h \rangle_h, && \forall v_h \in V_h^k, \\
\langle u_h, q_h \rangle_h &= 0, && \forall q_h \in \mathfrak{H}_h^k,
\end{aligned}
\tag{9}
$$

and the modified discrete problem

$$
\begin{aligned}
\langle J_h\sigma_h', \tau_h \rangle_h - \langle J_h u_h', \mathrm{d}_h\tau_h \rangle_h &= 0, && \forall \tau_h \in V_h^{k-1}, \\
\langle J_h\mathrm{d}_h\sigma_h', v_h \rangle_h + \langle J_h\mathrm{d}_h u_h', \mathrm{d}_h v_h \rangle_h + \langle J_h p_h', v_h \rangle_h &= \lambda_h' \langle u_h', v_h \rangle_h, && \forall v_h \in V_h^k, \\
\langle J_h u_h', q_h' \rangle_h &= 0, && \forall q_h' \in \mathfrak{H}_h'^k.
\end{aligned}
\tag{10}
$$

As shown by Arnold, Falk, and Winther [3, Theorem 3.19], solutions to the subcomplex problem (10) converge to those of (8), which follows immediately from the fact that $i_h K_h' P_h$ converges to K in the $\mathcal{L}\left(W^k, W^k\right)$ operator norm. We now show that this result also holds for the problem (9).

Theorem 3.13. *Let (V, d) be the domain complex of a closed Hilbert complex (W, d) satisfying the compactness property, and let (V_h, d_h) be a family of domain complexes of closed Hilbert complexes (W_h, d_h), equipped with morphisms $i_h \colon W_h \hookrightarrow W$ and $\pi_h \colon W \to W_h$ such that $\pi_h^k \circ i_h^k = \mathrm{id}_{W_h^k}$, where i_h and π_h are bounded uniformly with respect to h. Then the discrete eigenvalue problems (9) converge to the problem (8).*

Proof. It suffices to show that $i_h K_h P_h$ converges to K in the $\mathcal{L}\left(W^k, W^k\right)$ operator norm. (As stated by Arnold, Falk, and Winther [3], the sufficiency of norm convergence follows from Boffi, Brezzi, and Gastaldi [5].) Using the triangle inequality, we write

$$
\|K - i_h K_h P_h\| \leq \|K - i_h K_h' P_h\| + \|i_h (K_h' - K_h) P_h\|.
$$

The first term on the right-hand side converges to zero, by Arnold, Falk, and Winther [3, Corollary 3.17]. For the second term, recall that i_h and π_h are assumed to be bounded uniformly with respect to h, and since $\|P_h\| = \|\pi_h P_{i_h W_h}\| \leq \|\pi_h\|$, it follows that P_h is bounded uniformly with respect to h, as well. Therefore, it suffices to control $\|K_h' - K_h\|$ in $\mathcal{L}\left(W_h^k, W_h^k\right)$. However, the earlier analysis in

Theorem 3.10 shows that $\|K_h' - K_h\| \leq C \|I - J_h\|$, which completes the proof of convergence. $\square$

4. Application to differential forms on Riemannian manifolds

In this section, we apply the framework developed in Section 3 to the Hodge–de Rham complex of differential forms on a compact oriented Riemannian manifold. We will begin by first recalling the basic definitions of the de Rham complex of smooth forms; its completion as a Hilbert complex, called the L^2-de Rham complex; and the corresponding domain complex, which dovetails with the theory of Sobolev spaces. Next, we discuss the general problem of approximating the de Rham complex on a manifold M by a family of "nearby" manifolds M_h, each equipped with an orientation-preserving diffeomorphism $\varphi_h \colon M_h \to M$. We subsequently establish the correspondence between this setup and the generalized Hilbert complex approximation framework of Section 3, obtaining estimates for the appropriate maps, as needed. We then specialize the discussion a bit further by considering the case when M is a submanifold of some larger manifold N; in this case, the approximating submanifolds $M_h \subset N$ can be taken to lie in a tubular neighborhood of M, and $\varphi_h \colon M_h \to M$ is obtained by projection along normals.

Finally, we then look at the specific case where $N = \mathbb{R}^n$, and where we wish to approximate a solution on some m-dimensional Euclidean hypersurface $M \subset \mathbb{R}^n$, $n = m + 1$, by finite elements defined on a piecewise-linear mesh $M_h \subset \mathbb{R}^n$. This is now the realm of of surface finite element methods, as analyzed in Dziuk [16], Demlow and Dziuk [15], Demlow [14]. We subsequently show how our results of the previous sections recover the analysis framework and *a priori* estimates of Dziuk [16], Demlow and Dziuk [15], Demlow [14], extending their results from scalar functions on 2- and 3-surfaces to general k-forms on arbitrary dimensional hypersurfaces. We also indicate how our results generalize the *a priori* estimates of Dziuk [16], Demlow [14] from nodal finite element methods for the Laplace–Beltrami operator to mixed finite element methods for the Hodge Laplacian.

4.1. A brief review of Hodge–de Rham theory.

Given a smooth, m-dimensional manifold M, let $\Omega^k(M)$ denote the space of smooth k-forms on M for $k = 0, 1, \ldots, m$, and let $\mathrm{d}^k \colon \Omega^k(M) \to \Omega^{k+1}(M)$ be the exterior derivative for $k = 0, 1, \ldots, m-1$. Then $(\Omega(M), \mathrm{d})$ is a cochain complex,

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^m(M) \longrightarrow 0.$$

called the *de Rham complex* on M.

Suppose that, in addition, M is oriented and compact, and has a Riemannian metric g. Then, we can define the *L^2-inner product* of any $u, v \in \Omega^k(M)$ to be

$$\langle u, v \rangle_{L^2\Omega(M)} = \int_M u \wedge \star_g v = \int_M \langle\!\langle u, v \rangle\!\rangle_g \, \mu_g .$$

Here, $\star_g \colon \Omega^k(M) \to \Omega^{m-k}(M)$ is the Hodge star operator associated to the metric, $\langle\!\langle \cdot, \cdot \rangle\!\rangle_g \colon \Omega^k(M) \times \Omega^k(M) \to C^\infty(M)$ is the pointwise inner product induced by the metric and μ_g is the Riemannian volume form. (The Hodge star is defined precisely so that $u \wedge \star_g v = \langle\!\langle u, v \rangle\!\rangle_g \, \mu_g$, and it follows that $\star_g$ is an isometry.) The Hilbert space $L^2\Omega^k(M)$ is then defined, for each k, to be the completion of $\Omega^k(M)$ with respect to the L^2-inner product.

To show that this forms a Hilbert complex $\big(L^2\Omega(M), \mathrm{d}\big)$, we must now define the *weak exterior derivative* d^k on some dense domain of $L^2\Omega^k(M)$. Given $u \in L^2\Omega^k(M)$, we say that $w \in L^2\Omega^{k+1}(M)$ is the weak exterior derivative of u, and write $\mathrm{d}u = w$, if

$$\langle u, \mathrm{d}^* v\rangle_{L^2\Omega(M)} = \langle w, v\rangle_{L^2\Omega(M)}\,, \quad \forall v \in \Omega_c^{k+1}(M),$$

where $\Omega_c^{k+1}(M)$ denotes the space of smooth $(k+1)$-forms with compact support. Therefore, one defines the dense domains $H\Omega^k(M) \subset L^2\Omega^k(M)$, consisting of elements in $L^2\Omega^k(M)$ that have a weak exterior derivative in $L^2\Omega^{k+1}(M)$. Thus, we have

$$0 \longrightarrow H\Omega^0(M) \xrightarrow{\mathrm{d}} H\Omega^1(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} H\Omega^m(M) \longrightarrow 0.$$

where each $H\Omega^k(M)$ can be given the graph inner product

$$\langle u, v\rangle_{H\Omega(M)} = \langle u, v\rangle_{L^2\Omega(M)} + \langle \mathrm{d}u, \mathrm{d}v\rangle_{L^2\Omega(M)}\,.$$

(Note the similarity with the definition of the Sobolev spaces H^1, $H\,(\mathrm{curl})$, and $H\,(\mathrm{div})$.) Since each $H\Omega^k(M)$ is complete, it follows that d^k is a closed operator; therefore, $\big(L^2\Omega(M), \mathrm{d}\big)$ is indeed a Hilbert complex, and $(H\Omega(M), \mathrm{d})$ is the corresponding domain complex. Furthermore, it can be shown that $\big(L^2\Omega(M), \mathrm{d}\big)$ satisfies the compactness condition, so these Hilbert complexes are in fact closed and satisfy the conditions necessary for the improved error estimates. (For more details on the construction of these complexes, see Arnold, Falk, and Winther [3].)

4.2. **Diffeomorphic Riemannian manifolds.** Let (M, g) be an oriented, compact, m-dimensional Riemannian manifold, and suppose (M_h, g_h) is a family of oriented, compact Riemannian manifolds, parametrized by h and equipped with orientation-preserving diffeomorphisms $\varphi_h \colon M_h \to M$. Now, since the pullback $\varphi_h^* \colon \Omega(M) \to \Omega\,(M_h)$ and pushforward $\varphi_{h*} \colon \Omega\,(M_h) \to \Omega(M)$ commute with the exterior derivative, they give a cochain isomorphism between the smooth de Rham complexes $\Omega\,(M_h)$ and $\Omega\,(M)$.

We now show that these maps are bounded, and hence can be extended to Hilbert complex isomorphisms between $L^2\Omega\,(M_h)$ and $L^2\Omega(M)$, following the results of Stern [29]. Given any point $x \in M_h$, let $\{e_1, \dots, e_m\}$ be a positively-oriented, g_h-orthonormal basis of the tangent space $T_x M_h$, and let $\{f_1, \dots, f_m\}$ be a positively-oriented, g-orthonormal basis of $T_{\varphi_h(x)} M$. Then, with respect to these bases, the tangent map $T_x\varphi_h \colon T_x M_h \to T_{\varphi_h(x)} M$ can be represented by an $m \times m$ matrix Φ. Moreover, since φ_h is a diffeomorphism, the matrix Φ has m strictly positive singular values,

$$\alpha_1(x) \geq \cdots \geq \alpha_m(x) > 0.$$

These singular values are orthogonally invariant, so they are independent of the choice of basis at each x and $\varphi_h(x)$. Hence, they are an intrinsic property of the diffeomorphism, and thus we refer to them as the *singular values of φ_h at x.*

Theorem 4.1 (Stern [29], Corollary 6)**.** *Let (M_h, g_h) and (M, g) be oriented, m-dimensional Riemannian manifolds, and let $\varphi_h \colon M_h \to M$ be an orientation-preserving diffeomorphism with singular values $\alpha_1(x) \geq \cdots \geq \alpha_n(x) > 0$ at each $x \in M_h$. Given $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$, and some $k = 0, \dots, m$,* *suppose that the product* $(\alpha_1 \cdots \alpha_{m-k})^{1/p} (\alpha_{m-k+1} \cdots \alpha_m)^{-1/q}$ *is bounded uniformly on* M_h*. Then, for any* $\omega \in L^p\Omega^k(M_h)$*,*

$$\left\| (\alpha_1 \cdots \alpha_k)^{1/q} (\alpha_{k+1} \cdots \alpha_m)^{-1/p} \right\|_\infty^{-1} \|\omega\|_p \\ \leq \|\varphi_{h*}\omega\|_p \leq \left\| (\alpha_1 \cdots \alpha_{m-k})^{1/p} (\alpha_{m-k+1} \cdots \alpha_m)^{-1/q} \right\|_\infty \|\omega\|_p .$$

Sketch of proof. At each point, a k-form is k-linear and totally antisymmetric. Therefore, the pullback is controlled pointwise by the product of the k largest singular values of φ_h, while the pushforward is controlled by the product of the k largest singular values of φ_h^{-1} (i.e., the reciprocals of the k smallest singular values of φ_h). Thus, we obtain pointwise inequalities

$$|\varphi_h^*\eta| \leq \alpha_1 \cdots \alpha_k (|\eta| \circ \varphi_h), \qquad |\varphi_{h*}\omega| \leq \left[(\alpha_{m-k+1} \cdots \alpha_m)^{-1} |\omega| \right] \circ \varphi_h^{-1}.$$

For the L^p upper bound, we can apply the pushforward inequality to get a factor of $(\alpha_{m-k+1} \cdots \alpha_m)^{-p}$ in the integrand. Using the change of variables theorem introduces the Jacobian determinant $\alpha_1 \cdots \alpha_m$, so multiplying by this gives a factor of $\alpha_1 \cdots \alpha_{m-k} (\alpha_{m-k+1} \cdots \alpha_m)^{-p+1}$. We can then use Hölder's inequality to pull out the L^∞-norm of this expression, and raising to the exponent $1/p$ gives

$$\left\| (\alpha_1 \cdots \alpha_{m-k})^{1/p} (\alpha_{m-k+1} \cdots \alpha_m)^{-1+1/p} \right\|_\infty \\ = \left\| (\alpha_1 \cdots \alpha_{m-k})^{1/p} (\alpha_{m-k+1} \cdots \alpha_m)^{-1/q} \right\|_\infty,$$

as desired. The lower bound follows in a similar fashion, starting with the identity $\omega = \varphi_h^* \varphi_{h*}\omega$ and applying the pointwise pullback inequality. □

Since M and M_h are compact, the uniform boundedness hypothesis of this theorem is clearly satisfied. Therefore, taking $p = q = 2$, it follows that the diffeomorphism φ_h induces Hilbert complex isomorphisms $\varphi_{h*} \colon L^2\Omega(M_h) \to L^2\Omega(M)$ and $\varphi_h^* \colon L^2\Omega(M) \to L^2\Omega(M_h)$.

Now, take $W = L^2\Omega(M)$, and suppose that we have discrete subcomplexes $W_h \subset L^2\Omega(M_h)$ with inclusion morphisms $i_h' \colon W_h \hookrightarrow L^2\Omega(M_h)$, as well as projection morphisms $\pi_h' \colon L^2\Omega(M_h) \to W_h$ bounded uniformly in h. Following the approach of Theorem 3.7, we can pull these back to obtain the injection morphisms $i_h = \varphi_{h*} \circ i_h' \colon W_h \hookrightarrow W$ and projection morphisms $\pi_h = \pi_h' \circ \varphi_h^* \colon W \to W_h$, which satisfy $\pi_h \circ i_h = \mathrm{id}_{W_h}$. An important consequence of this is stated in the following corollary of Theorem 3.7 and Theorem 4.1.

Corollary 4.2. *Orientation-preserving diffeomorphisms induce an equivalence of families of finite element subcomplexes of the* L^2*-de Rham complex with bounded cochain projections. In particular, any triangulation* $\mathcal{T}_h \to M$ *gives corresponding* $\mathcal{P}_r^-$ *and* $\mathcal{P}_r$ *families (cf. Arnold, Falk, and Winther [2, 3]) of piecewise-polynomial differential forms on* M*.*

Finally, let us see how this definition of i_h can be used to control the error term $\|I - J_h\|$. Theorem 4.1 implies that, for any $v_h \in V_h^k$, we have the estimate

$$\left\| (\alpha_1 \cdots \alpha_k)^{1/2} (\alpha_{k+1} \cdots \alpha_m)^{-1/2} \right\|_\infty^{-1} \|v_h\|_h \\ \leq \|i_h v_h\| \leq \left\| (\alpha_1 \cdots \alpha_{m-k})^{1/2} (\alpha_{m-k+1} \cdots \alpha_m)^{-1/2} \right\|_\infty \|v_h\|_h ,$$

and since $J_h = i_h^* i_h$, this implies

$$\left\| \alpha_1 \cdots \alpha_k \left(\alpha_{k+1} \cdots \alpha_m \right)^{-1} \right\|_\infty^{-1} \left\| v_h \right\|_h \\ \leq \left\| J_h v_h \right\|_h \leq \left\| \alpha_1 \cdots \alpha_{m-k} \left(\alpha_{m-k+1} \cdots \alpha_m \right)^{-1} \right\|_\infty \left\| v_h \right\|_h .$$

This bounds the spectrum of the self-adjoint operator J_h, so finally we obtain a bound on the error term $\| I - J_h \|$ in terms of the singular values,

$$\| I - J_h \| \leq \max \Big\{ \left| 1 - \left\| \alpha_1 \cdots \alpha_k \left(\alpha_{k+1} \cdots \alpha_m \right)^{-1} \right\|_\infty^{-1} \right| , \\ \left| 1 - \left\| \alpha_1 \cdots \alpha_{m-k} \left(\alpha_{m-k+1} \cdots \alpha_m \right)^{-1} \right\|_\infty \right| \Big\}. \tag{11}$$

It follows that, if each singular value satisfies $|1 - \alpha_i| \leq C h^{s+1}$, then $\| I - J_h \| \leq C h^{s+1}$ as well, and moreover this will hold for every $k = 0, \ldots, m$. Obtaining such bounds on the singular values, for particular choices of φ_h, will be the topic of the next subsection.

4.3. **Tubular neighborhoods and Euclidean hypersurfaces.** Suppose that (N, γ) is an oriented, n-dimensional Riemannian manifold, and let $j \colon M \hookrightarrow N$ be the inclusion of a submanifold M, endowed with the metric $g = j^* \gamma$ inherited from N. If M is compact, then it is possible to construct a *tubular neighborhood* U around M; this is diffeomorphic to an open neighborhood of the zero section of the normal bundle of M, so there is a normal projection map $a \colon U \to M$. In particular, there exists some $\delta_0 > 0$ such that the set M_{δ_0}, consisting of points in N whose Riemannian distance to M is less than δ_0, is contained in U. (For details, see, e.g., Abraham and Marsden [1], Lang [23], Lee [24].) Now, let $j_h \colon M_h \hookrightarrow N$ be a family of inclusions of m-dimensional submanifolds M_h, parametrized by h, each endowed with the Riemannian metric $g_h = j_h^* \gamma$. If M_h lies inside the tubular neighborhood U and is transverse to a (i.e., M_h corresponds to a section of a), then it is possible to define the diffeomorphism $\varphi_h = a|_{M_h} \colon M_h \to M$.

An important case is when $N = \mathbb{R}^n$, where $n = m + 1$ and γ is the standard Euclidean metric, so that $M \subset \mathbb{R}^n$ is an oriented Euclidean hypersurface. It is possible to define a signed distance function $\delta \colon U \to \mathbb{R}$ on the tubular neighborhood, so that $|\delta(x)| = \operatorname{dist}(x, M)$ and $\nabla \delta(x) = \nu(x)$ is the outward-facing unit normal to M at $a(x)$. Every point $x \in U$ in the tubular neighborhood has a unique decomposition

$$x = a(x) + \delta(x) \nu(x),$$

so the normal projection map $a \colon U \to M$ can be written as

$$a(x) = x - \delta(x) \nu(x).$$

Therefore,

$$\nabla a = I - \nabla \delta \otimes \nu - \delta \nabla \nu = I - \nu \otimes \nu - \delta \nabla \nu = P + \delta S,$$

where $P = I - \nu \otimes \nu$ is the projection map onto TM and $S = -\nabla \nu = -\nabla^2 \delta$ is the shape operator, or Weingarten map. (Note that Dziuk [16], Demlow and Dziuk [15], Demlow [14] define a Weingarten map $H = -S$ using the opposite sign convention, but this is less common in the differential geometry literature.)

Instead of directly computing the tangent map $Ta \colon U \to M$, we can look at its adjoint, which "lifts" vectors on M to those on U. Given the pullback map $a^* \colon \Omega^1(M) \to \Omega^1(U)$, the metric can then be used to identify covectors with vectors, thereby obtaining a pullback map of vector fields $\mathfrak{X}(M) \to \mathfrak{X}(U)$. Specifically, let $Y \in T_y M$ and $x \in a^{-1}(y) \subset U$. Then define the lifted vector $a^* Y \in T_x U$ satisfying

$$X \cdot a^* Y = Ta(X) \cdot Y, \quad \forall X \in T_x U.$$

In terms of the Riemannian sharp and flat maps, this can be written as

$$[a^*(Y)]^\flat = a^*\left(Y^\flat\right) \Longleftrightarrow a^* Y = \left[a^*\left(Y^\flat\right)\right]^\sharp.$$

The following theorem allows us to compute this lifted vector explicitly, in terms of the signed distance function and shape operator.

Theorem 4.3. *Let M be an oriented, compact, m-dimensional hypersurface of $\mathbb{R}^{m+1}$ with a tubular neighborhood U. If $Y \in T_y M$ and $x \in a^{-1}(y) \subset U$, then the lifted vector $a^* Y \in T_x U$ satisfies*

$$a^* Y = (I + \delta S)\, Y.$$

Proof. Extend Y to a constant vector field on $\mathbb{R}^{m+1}$, so that $Y = \nabla \psi(y)$ for the scalar function $\psi(x) = x \cdot Y$. Using the definition of the gradient $\nabla \psi = (\mathrm{d}\psi)^\sharp$, and the fact that the exterior derivative d commutes with pullback, we have the following chain of equalities:

$$a^* Y = a^*\left(\nabla \psi\right) = \left[a^*\left(\mathrm{d}\psi\right)\right]^\sharp = \left[\mathrm{d}\left(a^*\psi\right)\right]^\sharp = \nabla\left(\psi \circ a\right).$$

Therefore, applying the chain rule, we get

$$a^* Y = \nabla a(x) \cdot \nabla \psi\left(a(x)\right) = (P + \delta S)\, Y = (I + \delta S)\, Y,$$

where the last equality follows from $PY = Y$. □

Finally, when $x \in M_h$, we can restrict to $T_x M_h$ by composing with the adjoint of j_h, i.e., the projection $P_h = I - \nu_h \otimes \nu_h$, which gives

$$Y_h = j_h^* a^* Y = P_h\,(I + \delta S)\, Y.$$

This map $j_h^* a^* = P_h\,(I + \delta S)$ is immediately seen to be the adjoint of the restricted tangent map $T\varphi_h = T\left(a|_{M_h}\right) = T\left(a \circ j_h\right) = Ta \circ Tj_h$. In the next theorem, we bound the singular values of this map, thereby obtaining an estimate for the error term $\lVert I - J_h \rVert$ in the case of Euclidean hypersurfaces.

Theorem 4.4. *Given an oriented, compact, m-dimensional hypersurface $M \subset \mathbb{R}^{m+1}$ with a tubular neighborhood U, let M_h be a family of hypersurfaces lying in U and transverse to its fibers, such that $\lVert\delta\rVert_\infty, \lVert\nu - \nu_h\rVert_\infty \to 0$ as $h \to 0$. Then, for sufficiently small h,*

$$\lVert I - J_h \rVert \leq C\left(\lVert\delta\rVert_\infty + \lVert\nu - \nu_h\rVert_\infty^2\right).$$

Proof. To obtain bounds on $Y_h = P_h a^* Y$, and hence on the singular values, suppose without loss of generality that $\lvert Y \rvert = 1$. By the triangle inequality,

$$\left|1 - \lvert Y_h\rvert^2\right| \leq \left|1 - \lvert a^* Y\rvert^2\right| + \left|\lvert a^* Y\rvert^2 - \lvert Y_h\rvert^2\right|.$$

For the first term, the eigenvalues of the shape operator are the principal curvatures $\kappa_1\left(x\right), \ldots, \kappa_m(x)$ for a surface parallel to M at x; as noted in Demlow and Dziuk [15], Demlow [14], these are related to the principal curvatures at $a(x) \in M$ by

$$\kappa_i(x) = \frac{\kappa_i\left(a(x)\right)}{1 + \delta(x)\kappa_i\left(a(x)\right)}.$$

It follows that the eigenvalues of δS at x can be estimated by
$$\left| \frac{\delta(x)\kappa_i\left(a(x)\right)}{1+\delta(x)\kappa_i\left(a(x)\right)} \right| = \left| 1 - \frac{1}{1+\delta(x)\kappa_i\left(a(x)\right)} \right| \leq C \left|\delta(x)\right| .$$
Since $a^*Y = \left(I + \delta S\right) Y$ and $\lvert Y \rvert = 1$, this immediately implies
$$\left| 1 - \left|a^*Y\right|^2 \right| \leq C \left|\delta\right| .$$
For the remaining term, observe that since $Y_h = P_h a^* Y$,
$$\left|Y_h\right|^2 = \left|a^*Y - \nu_h \left(\nu_h \cdot a^*Y\right)\right|^2 = \left|a^*Y\right|^2 - \left(\nu_h \cdot a^*Y\right)^2 ,$$
and therefore
$$\left| \left|a^*Y\right|^2 - \left|Y_h\right|^2 \right| = \left(\nu_h \cdot a^*Y\right)^2 = \left(P\nu_h \cdot a^*Y\right)^2 \leq \left|P\nu_h\right|^2 \left|a^*Y\right|^2 .$$
Now,
$$\left|P\nu_h\right|^2 = \left|\nu_h - \nu\left(\nu\cdot\nu_h\right)\right|^2 = 1 - \left(\nu\cdot\nu_h\right)^2 = \left(1+\nu\cdot\nu_h\right)\left(1-\nu\cdot\nu_h\right) \leq 2\left(1-\nu\cdot\nu_h\right),$$
and since $2\left(1-\nu\cdot\nu_h\right) = \left|\nu-\nu_h\right|^2$, it follows that
$$\left| \left|a^*Y\right|^2 - \left|Y_h\right|^2 \right| \leq \left|\nu-\nu_h\right|^2 \left|a^*Y\right|^2 \leq \left|\nu-\nu_h\right|^2 \left(1 + \left| 1 - \left|a^*Y\right|^2 \right| \right) \leq C \left|\nu-\nu_h\right|^2 .$$
Putting these together, we have
$$\left| 1 - \left|Y_h\right|^2 \right| \leq C \left(\left|\delta\right| + \left|\nu-\nu_h\right|^2 \right),$$
from which it follows that at each $x \in M_h$, the singular values satisfy
$$\left|1-\alpha_i\right| \leq C \left(\left|\delta\right| + \left|\nu-\nu_h\right|^2 \right), \quad i = 1, \ldots, m.$$
Finally, applying (11), we obtain the uniform bound
$$\left\lVert I - J_h \right\rVert \leq C \left(\left\lVert\delta\right\rVert_\infty + \left\lVert\nu-\nu_h\right\rVert_\infty^2 \right),$$
which completes the proof. □

We now apply this theory to an important class of examples, where M_h corresponds to a family of piecewise-linear triangulations (as in Dziuk [16], Demlow and Dziuk [15]), or more generally, to the family of approximate surfaces obtained by degree-s Lagrange interpolation over each element of the triangulation (as in Demlow [14]), where the piecewise-linear case corresponds to $s = 1$. Here, the elements of this triangulation are assumed to be "shape-regular and quasi-uniform of diameter h" [14]. Note that M_h is always constructed from an underlying piecewise-linear triangulation, even in the case of higher-order polynomial interpolation. Thus, by Corollary 4.2, we can define the $\mathcal{P}_r^-$ and $\mathcal{P}_r$ families of finite element differential forms on M_h, and obtain bounded cochain projections, even when $s > 1$.

By Demlow [14, Proposition 2.3], for sufficiently small h, the surfaces M_h obtained by degree-s Lagrange interpolation satisfy
$$\left\lVert\delta\right\rVert_\infty \leq C h^{s+1}, \qquad \left\lVert\nu-\nu_h\right\rVert_\infty \leq C h^s. \tag{12}$$
Therefore, we obtain the following corollary to Theorem 4.4.

Corollary 4.5. *If M_h is a family of surfaces approximating M, obtained by degree-s Lagrange interpolation, then $\left\lVert I - J_h\right\rVert \leq C h^{s+1}$.*

Proof. Applying (12), we have

$$\lVert I - J_h \rVert \leq C \left(\lVert \delta \rVert_\infty + \lVert \nu - \nu_h \rVert_\infty^2 \right) \leq C h^{s+1} + C h^{2s} \leq C h^{s+1},$$

which completes the proof. □

This result generalizes Demlow [14, Proposition 4.1]—which applies only to scalar functions ($k = 0$) on hypersurfaces of dimension $m = 2, 3$—to hold for arbitrary k-forms, $k = 0, \ldots, m$, on hypersurfaces of any dimension. In particular, the special case $k = 0$, $m = 2$, $s = 1$, gives $\lVert I - J_h \rVert \leq C h^2$, which recovers the original estimate of Dziuk [16] for piecewise-linear triangulation of surfaces in $\mathbb{R}^3$. The correspondence between this framework, and that of Dziuk and Demlow, will be made explicit in the following worked example.

Example 4.6 (The Hodge–Laplace operator on a 2-D surface)**.** Let M be a closed, two-dimensional surface, embedded in $\mathbb{R}^3$, and suppose the approximate surface M_h is obtained from degree-s Lagrange interpolation over a piecewise-linear triangulation $\mathcal{T}_h$. Assume that $\mathcal{T}_h$ is contained in a tubular neighborhood of M, that its vertices lie on M, and that its triangles are shape-regular and quasi-uniform of diameter h.

Take the continuous Hilbert complex to be the L^2-de Rham complex on M, i.e., $W = L^2\Omega(M)$ and $V = H\Omega(M)$. Since $\mathcal{T}_h$ is piecewise-linear and simplicial, we can take the discrete complex to be any of those considered in Arnold, Falk, and Winther [2, 3]. For this example, let us take V_h^k to be the space of $\mathcal{P}_r$ k-forms, and V_h^{k-1} to be the space of $\mathcal{P}_{r+1}$ $(k-1)$-forms. We emphasize that the fact that $\mathcal{T}_h$ is a surface embedded in $\mathbb{R}^3$, rather than a flat region in $\mathbb{R}^2$, does not introduce any additional complications as far as the discrete complex is concerned. Indeed, the shape functions are defined with respect to a two-dimensional reference triangle, and this reference triangle can be mapped onto a triangle embedded in $\mathbb{R}^3$ just as easily as one in $\mathbb{R}^2$. These shape functions can, likewise, be lifted up from $\mathcal{T}_h$ to the curved triangles on the interpolated surface M_h. For nodal Lagrange finite elements ($k = 0$), this observation was made by Dziuk [16] in the piecewise linear case, leading to the development of surface finite elements, while Demlow [14] later extended this argument to higher-order Lagrange polynomials. (Similar ideas had also been used in the development of isoparametric finite elements for Euclidean domains with curved boundaries.)

Now, given some $f \in L^2\Omega^k(M)$, we obtain a solution (σ, u, p) to the variational problem (3) on M. For the discrete variational problem (5), we can use the tubular neighborhood projection to take $f_h = a|_{M_h}^* f$, thus obtaining a discrete solution (σ_h, u_h, p_h) on M_h. The modified discrete solution (σ_h', u_h', p_h')—which is used only for the analysis, but is not necessary for computation—also lives on M_h, while its image $(i_h \sigma_h', i_h u_h', i_h p_h')$ lives on M itself.

Therefore, assuming sufficient elliptic regularity, the "improved estimates" of Arnold, Falk, and Winther [2, 3] yield the L^2 estimates for the modified problem,

$$\begin{aligned}
\lVert u - i_h u_h' \rVert + \lVert p - i_h p_h' \rVert &\leq C h^{r+1} \lVert f \rVert_{H^{r-1}}, \\
\lVert \mathrm{d}\,(u - i_h u_h') \rVert + \lVert \sigma - i_h \sigma_h' \rVert &\leq C h^r \lVert f \rVert_{H^{r-1}}, \\
\lVert \mathrm{d}\,(\sigma - i_h \sigma_h') \rVert &\leq C h^{r-1} \lVert f \rVert_{H^{r-1}},
\end{aligned}$$

which can be combined into the single estimate

$$\|u - i_h u_h'\| + \|p - i_h p_h'\| + h\left(\|\mathrm{d}\,(u - i_h u_h')\| + \|\sigma - i_h \sigma_h'\|\right) \\ + h^2 \left\|\mathrm{d}\,(\sigma - i_h \sigma_h')\right\| \leq C h^{r+1} \|f\|_{H^{r-1}} .$$

Applying Corollary 4.5 to account for the surface approximation error, we obtain the final error estimate for the discrete problem,

$$\|u - i_h u_h\| + \|p - i_h p_h\| + h\left(\|\mathrm{d}\,(u - i_h u_h)\| + \|\sigma - i_h \sigma_h\|\right) \\ + h^2 \left\|\mathrm{d}\,(\sigma - i_h \sigma_h)\right\| \leq C\left(h^{r+1} \|f\|_{H^{r-1}} + h^{s+1}\|f\|\right).$$

In particular, this implies that choosing *isoparametric elements*, with $r = s$, yields the optimal rate of convergence.

The case $k = 0$ and $r = s = 1$ corresponds to the lowest-order approximation of the Laplace–Beltrami equation, where M_h is piecewise-linear and V_h^0 consists of piecewise-linear "hat functions" on M_h. In this case, the estimate above becomes

$$\|u - i_h u_h\| + \|p - i_h p_h\| + h\left\|\nabla\,(u - i_h u_h)\right\| \leq C h^2 \|f\|,$$

which precisely recovers the estimate of Dziuk [16], Demlow and Dziuk [15]. More generally, taking $k = 0$ with arbitrary r and s, we have

$$\|u - i_h u_h\| + \|p - i_h p_h\| + h\left\|\nabla\,(u - i_h u_h)\right\| \leq C\left(h^{r+1} \|f\|_{H^{r-1}} + h^{s+1}\|f\|\right),$$

which agrees with Demlow [14].

On the other hand, we can also extend these estimates to the cases $k = 1$, which corresponds to the mixed formulation of the vector Laplacian, and $k = 2$, which corresponds to the mixed formulation of the scalar Laplacian. For $k = 1$, the estimate for general r and s becomes

$$\|u - i_h u_h\| + \|p - i_h p_h\| + h\left(\|\nabla \times (u - i_h u_h)\| + \|\sigma - i_h \sigma_h\|\right) \\ + h^2 \left\|\nabla\,(\sigma - i_h \sigma_h)\right\| \leq C\left(h^{r+1} \|f\|_{H^{r-1}} + h^{s+1}\|f\|\right),$$

while for $k = 2$, we obtain

$$\|u - i_h u_h\| + \|p - i_h p_h\| + h\left\|\sigma - i_h \sigma_h\right\| \\ + h^2 \left\|\nabla \cdot (\sigma - i_h \sigma_h)\right\| \leq C\left(h^{r+1} \|f\|_{H^{r-1}} + h^{s+1}\|f\|\right).$$

4.4. **Other variational crimes.** The variational crimes framework developed in Section 3 is quite general, representing a natural extension of the Strang lemmas from Hilbert spaces to Hilbert complexes. As such, the standard "crimes" that are typically analyzed in Hilbert spaces with the Strang lemmas may now be analyzed in the setting of Hilbert complexes. These crimes—including numerical quadrature, approximate coefficients, approximate boundary data, approximate domains, as well as isoparametric and other geometric approximations to more complex domain shapes—can all be represented as an approximate bilinear form B_h, an approximate linear functional F_h, and an approximation subspace $V_h \not\subset V$, as in (2). In addition, techniques such as *mass-lumping*, which yield a number of benefits, such as discrete maximum principles and more efficient evolution algorithms for parabolic equations, are often analyzed in a similar way, and as such may now be analyzed in Hilbert complexes through the framework developed in Section 3.

5. Conclusion

We began the article in Section 2 with a review of the mathematical concepts that play a fundamental role in finite element exterior calculus, as developed in Arnold, Falk, and Winther [3]; these included abstract Hilbert complexes and their morphisms, domain complexes, Hodge decomposition, the Poincaré inequality, the Hodge Laplacian, mixed variational problems, and approximation using Hilbert subcomplexes. In Section 3, we then considered approximation of a Hilbert complex by a second complex, related to the first complex through an injective morphism rather than through subcomplex inclusion. We developed several key results for this pair of complexes and the maps between them, and then derived error estimates for generalized Galerkin-type approximations of solutions to variational problems using the approximating complex; these estimates can be viewed as generalizing the results of Arnold, Falk, and Winther [3] to "external" approximations. Our main abstract results are thus essentially *Strang-type lemmas* for approximating variational problems in Hilbert complexes.

As an application of the new framework developed in Section 3, we developed a second distinct set of results in Section 4 for the case of the Hodge–de Rham complex of differential forms on a compact, oriented Riemannian manifold. We first reviewed Hodge–de Rham theory, and then considered a pair of Riemannian manifolds related by diffeomorphisms. We then established estimates for the maps needed to apply the generalized Hilbert complex approximation framework from Section 3, subsequently specializing this analysis to the case of a Euclidean hypersurface, with approximating hypersurfaces living in a tubular neighborhood. The surface finite element methods, as analyzed in Dziuk [16], Demlow and Dziuk [15], Demlow [14], fit precisely into this class of approximation problems; as such, we illustrated how our results recover the analysis framework and *a priori* estimates of Dziuk [16], Demlow and Dziuk [15], Demlow [14], and also extend their results from scalar functions on 2- and 3-surfaces to general k-forms on arbitrary dimensional hypersurfaces. Our results also generalize those earlier estimates from nodal finite element methods for the Laplace–Beltrami operator to mixed finite element methods for the Hodge Laplacian. By analyzing surface finite element methods using a combination of general tools from differential geometry and functional analysis, we are led to a more geometric analysis of surface finite element methods, whereby the main results become more transparent.

There remain a number of interesting and challenging problems that were not addressed in the current article. One such problem is the extension of the pointwise error estimates of Demlow [14] for 0-forms to general k-forms; this analysis relies on known results for the Green's function of the Laplace–Beltrami operator on the continuous surface (cf. [4]), and analogous results would be needed for general k-forms. A second problem of interest is an extension of the Hilbert complex framework to more general Banach complexes, as would be needed to handle some nonlinear problems. This leads to a third interesting problem, which would involve the extension of the weak-$*$ convergence and contraction frameworks, used for adaptive finite element methods for linear [9, 25] and nonlinear [22] problems, to the setting of finite element exterior calculus, as well as to the surface finite element setting.

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