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SEMILINEAR MIXED PROBLEMS ON HILBERT COMPLEXES AND THEIR NUMERICAL APPROXIMATION

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ABSTRACT. Arnold, Falk, and Winther recently showed [*Bull. Amer. Math. Soc.* **47** (2010), 281–354] that linear, mixed variational problems, and their numerical approximation by mixed finite element methods, can be studied using the powerful, abstract language of Hilbert complexes. In another recent article [arXiv:1005.4455], we extended the Arnold–Falk–Winther framework by analyzing variational crimes (a la Strang) on Hilbert complexes. In particular, this gave a treatment of finite element exterior calculus on manifolds, generalizing techniques from surface finite element methods and recovering earlier *a priori* estimates for the Laplace–Beltrami operator on 2- and 3-surfaces, due to Dziuk [*Lecture Notes in Math.*, vol. 1357 (1988), 142–155] and later Demlow [*SIAM J. Numer. Anal.*, **47** (2009), 805–827], as special cases. In the present article, we extend the Hilbert complex framework in a second distinct direction: to the study of semilinear mixed problems. We do this, first, by introducing an operator-theoretic reformulation of the linear mixed problem, so that the semilinear problem can be expressed as an abstract Hammerstein equation. This allows us to obtain, for semilinear problems, *a priori* solution estimates and error estimates that reduce to the Arnold–Falk–Winther results in the linear case. We also consider the impact of variational crimes, extending the results of our previous article to these semilinear problems. As an immediate application, this new framework allows for mixed finite element methods to be applied to semilinear problems on surfaces.

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Contents

1. Introduction

The goal of this paper is to extend the abstract Hilbert complex framework of Arnold, Falk, and Winther [4]—which they introduced to analyze certain linear mixed variational problems and their numerical approximation by mixed finite elements—to a class of *semilinear* mixed variational problems. Additionally, we aim to analyze variational crimes in this semilinear setting, extending our earlier analysis of the linear case in Holst and Stern [24].

1.1. **Background.** Brüning and Lesch [9] originally studied Hilbert complexes as a way to generalize certain properties of elliptic complexes, particularly the Hodge decomposition and other aspects of Hodge theory. More recently, Arnold, Falk, and Winther [4] showed that Hilbert complexes are also a convenient abstract setting for mixed variational problems and their numerical approximation by mixed finite element methods, providing the foundation of a framework called *finite element exterior calculus* (see also [3]). This line of research is the culmination of several decades of work on mixed finite element methods, which have long been used with great success in computational electromagnetics, and which were more recently discovered to have surprising connections with the calculus of exterior differential forms, including de Rham cohomology and Hodge theory [6, 27, 28, 21]. For this reason, Hilbert complexes are a natural fit for abstract methods of this type.

Another recent development in this area has been the analysis of "variational crimes" on Hilbert complexes (Holst and Stern [24]). By analogy with Strang's lemmas for variational crimes on Hilbert spaces, this work extended the estimates of Arnold, Falk, and Winther [4] to problems where certain conditions on the discretization have been violated. This framework also allowed for a generalization of several results in the field of *surface finite element methods*, where a curved domain is not triangulated exactly, but is only approximated by, e.g., piecewise linear or isoparametric elements. This research area was initiated with the 1988 article of Dziuk [17] (see also Nédélec [26]), with growing activity in the 1990s [18, 12] and a substantial expansion beginning around 2001 [22, 11, 13, 14, 20, 19, 16, 15].

Our main motivation for extending the estimates of Arnold, Falk, and Winther [4] and of Holst and Stern [24], from linear to semilinear problems, is to enable the use of finite element exterior calculus for nonlinear problems on hypersurfaces, allowing for a complete analysis of the additional errors due to nonlinearity, as well as those due to surface approximation.

1.2. **Organization of the paper.**

The remainder of the article is structured as follows. In Section 2 we give a quick overview of abstract Hilbert complexes and their properties, before introducing the Hodge Laplacian and the linear mixed problem associated with it. We then discuss the numerical approximation of solutions to this problem, summarizing some of the key results of Arnold, Falk, and Winther [4] on approximation by subcomplexes, and those of Holst and Stern [24] on variational crimes. In Section 3, we introduce an alternative, operator-theoretic formalism for the linear problem, which—while equivalent to the mixed variational formulation—allows for a more natural extension to semilinear problems, due to its monotonicity properties. We then introduce a class of semilinear problems—which can be expressed in the form of certain nonlinear operator equations, called abstract Hammerstein equations—prove the well-posedness of these problems, and establish solution estimates under various assumptions on the nonlinear part. In Section 4, we extend the *a priori* error estimates of Arnold, Falk, and Winther [4] from linear problems to the semilinear problems introduced in Section 3, including improved estimates subject to additional compactness and continuity assumptions. Finally, we generalize the linear variational crimes framework of [24] to this class of semilinear problems. These last results allow the linear *a priori* estimates, established in [24] for surface finite element methods using differential forms on hypersurfaces, to be extended to semilinear problems.

2. Review of Hilbert complexes and linear mixed problems

We begin, in this section, by quickly recalling the basic objects of interest—Hilbert complexes and the abstract Hodge Laplacian—along with the solution theory for linear mixed problems in this setting. This provides the background and preparation for semilinear problems, which will be discussed in the subsequent sections. The treatment of this background material will be necessarily brief; we will primarily follow the approach of Arnold, Falk, and Winther [4], to which the interested reader should refer for more detail.[1] At the end of the section, we will also summarize

[1]This is largely a condensed version of the background material given in Holst and Stern [24, Section 2], from which we quote freely. We include it here in the interest of keeping the present paper self-contained, since the semilinear theory will depend, to a large degree, on several properties and results that have recently been established for the linear problem.

the results from Holst and Stern [24], analyzing variational crimes for the linear problem, in preparation for extending these results to the semilinear case.

2.1. **Basic definitions.** First, let us introduce the objects of study, Hilbert complexes, and their morphisms.

Definition 2.1. A *Hilbert complex* (W, d) consists of a sequence of Hilbert spaces W^k, along with closed, densely-defined linear maps $\mathrm{d}^k \colon V^k \subset W^k \to V^{k+1} \subset W^{k+1}$, possibly unbounded, such that $\mathrm{d}^k \circ \mathrm{d}^{k-1} = 0$ for each k.

$$\cdots \longrightarrow V^{k-1} \xrightarrow{\mathrm{d}^{k-1}} V^k \xrightarrow{\mathrm{d}^k} V^{k+1} \longrightarrow \cdots$$

This Hilbert complex is said to be *bounded* if d^k is a bounded linear map from W^k to W^{k+1} for each k, i.e., (W, d) is a cochain complex in the category of Hilbert spaces. It is said to be *closed* if the image $\mathrm{d}^k V^k$ is closed in W^{k+1} for each k.

Definition 2.2. Given two Hilbert complexes, (W, d) and (W', d'), a *morphism of Hilbert complexes* $f \colon W \to W'$ consists of a sequence of bounded linear maps $f^k \colon W^k \to W'^k$ such that $f^k V^k \subset V'^k$ and $\mathrm{d}'^k f^k = f^{k+1} \mathrm{d}^k$ for each k. That is, the following diagram commutes:

$$\begin{array}{ccccccc} \cdots \longrightarrow & V^k & \xrightarrow{\mathrm{d}^k} & V^{k+1} & \longrightarrow \cdots \\ & \big\downarrow f^k & & \big\downarrow f^{k+1} & \\ \cdots \longrightarrow & V'^k & \xrightarrow{\mathrm{d}'^k} & V'^{k+1} & \longrightarrow \cdots \end{array}$$

By analogy with cochain complexes, it is possible to define notions of cocycles, coboundaries, and harmonic forms for Hilbert complexes. (This also gives rise to a cohomology theory for Hilbert complexes.)

Definition 2.3. Given a Hilbert complex (W, d), the space of *k-cocycles* is the kernel $\mathfrak{Z}^k = \ker \mathrm{d}^k$, the space of *$k$-coboundaries* is the image $\mathfrak{B}^k = \mathrm{d}^{k-1} V^{k-1}$, and the *$k$th harmonic space* is the intersection $\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{B}^{k\perp}$.

In general, the differentials d^k of a Hilbert complex may be unbounded linear maps. However, given an arbitrary Hilbert complex (W, d), it is always possible to construct a bounded complex having the same domains and maps, as follows.

Definition 2.4. Given a Hilbert complex (W, d), the *domain complex* (V, d) consists of the domains $V^k \subset W^k$, endowed with the graph inner product

$$\langle u, v \rangle_{V^k} = \langle u, v \rangle_{W^k} + \langle \mathrm{d}^k u, \mathrm{d}^k v \rangle_{W^{k+1}}.$$

Remark 1. Since d^k is a closed map, each V^k is closed with respect to the norm induced by the graph inner product. Also, each map d^k is bounded, since

$$\lVert \mathrm{d}^k v \rVert_{V^{k+1}} = \lVert \mathrm{d}^k v \rVert_{W^{k+1}} \leq \lVert v \rVert_{W^k} + \lVert \mathrm{d}^k v \rVert_{W^{k+1}} = \lVert v \rVert_{V^k}.$$

Thus, the domain complex is a bounded Hilbert complex; moreover, it is a closed complex if and only if (W, d) is closed.

Example 2.5. Perhaps the most important example of a Hilbert complex arises from the de Rham complex $(\Omega(M), \mathrm{d})$ of smooth differential forms on an oriented, compact, Riemannian manifold M, where d is the exterior derivative. Given two smooth k-forms $u, v \in \Omega^k(M)$, the L^2-inner product is defined by

$$\langle u, v \rangle_{L^2\Omega(M)} = \int_M u \wedge \star v = \int_M \langle\!\langle u, v \rangle\!\rangle \, \mu,$$

where $\star$ is the Hodge star operator associated to the Riemannian metric, $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is the metric itself, and μ is the Riemannian volume form. The Hilbert space $L^2\Omega^k(M)$ is then defined, for each k, to be the completion of $\Omega^k(M)$ with respect to the L^2-inner product. One can also define weak exterior derivatives $\mathrm{d}^k \colon H\Omega^k(M) \subset L^2\Omega^k(M) \to H\Omega^{k+1}(M) \subset L^2\Omega^{k+1}(M)$; the domain complex $(H\Omega(M), \mathrm{d})$, with the graph inner product

$$\langle u, v \rangle_{H\Omega(M)} = \langle u, v \rangle_{L^2\Omega(M)} + \langle \mathrm{d}u, \mathrm{d}v \rangle_{L^2\Omega(M)},$$

is analogous to a Sobolev space of differential forms. (For example, in $\mathbb{R}^3$, the domain complex corresponds to the spaces H^1, $H(\mathrm{curl})$, and $H(\mathrm{div})$.) Finally, we mention the fact that both the L^2- and H-de Rham complexes are closed. For a detailed treatment of these complexes, and their many applications, see Arnold, Falk, and Winther [4].

For the remainder of the paper, we will follow the simplified notation used by Arnold, Falk, and Winther [4]: the W-inner product and norm will be written simply as $\langle \cdot, \cdot \rangle$ and $\lVert \cdot \rVert$, without subscripts, while the V-inner product and norm will be written explicitly as $\langle \cdot, \cdot \rangle_V$ and $\lVert \cdot \rVert_V$.

2.2. **Hodge decomposition and the Poincaré inequality.** For L^2 differential forms, the Hodge decomposition states that any k-form can be written as a direct sum of exact, coexact, and harmonic components. (In $\mathbb{R}^3$, this corresponds to the Helmholtz decomposition of vector fields.) In fact, this can be generalized to give a Hodge decomposition for arbitrary Hilbert complexes; this immediately gives rise to an abstract version of the Poincaré inequality, which is crucial to much of the analysis in Arnold, Falk, and Winther [4].

Following Brüning and Lesch [9], we can decompose each space W^k in terms of orthogonal subspaces,

$$W^k = \mathfrak{Z}^k \oplus \mathfrak{Z}^{k\perp_W} = \mathfrak{Z}^k \cap \left(\overline{\mathfrak{B}^k} \oplus \mathfrak{B}^{k\perp} \right) \oplus \mathfrak{Z}^{k\perp_W} = \overline{\mathfrak{B}^k} \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_W},$$

where the final expression is known as the *weak Hodge decomposition*. For the domain complex (V, d), the spaces $\mathfrak{Z}^k$, $\mathfrak{B}^k$, and $\mathfrak{H}^k$ are the same as for (W, d), and consequently we get the decomposition

$$V^k = \overline{\mathfrak{B}^k} \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_V},$$

where $\mathfrak{Z}^{k\perp_V} = \mathfrak{Z}^{k\perp_W} \cap V^k$. In particular, if (W, d) is a closed Hilbert complex, then the image $\mathfrak{B}^k$ is a closed subspace, so we have the *strong Hodge decomposition*

$$W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_W},$$

and likewise for the domain complex,

$$V^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_V}.$$

From here on, following the notation of Arnold, Falk, and Winther [4], we will simply write $\mathfrak{Z}^{k\perp}$ in place of $\mathfrak{Z}^{k\perp_V}$ when there can be no confusion.

Lemma 2.6 (abstract Poincaré inequality)**.** *If* (V, d) *is a bounded, closed Hilbert complex, then there exists a constant* c_P *such that*

$$\|v\|_V \leq c_P \|\mathrm{d}^k v\|_V, \quad \forall v \in \mathfrak{Z}^{k\perp}.$$

Proof. The map d^k is a bounded bijection from $\mathfrak{Z}^{k\perp}$ to $\mathfrak{B}^{k+1}$, which are both closed subspaces, so the result follows immediately by applying Banach's bounded inverse theorem. □

Corollary 2.7. *If* (V, d) *is the domain complex of a closed Hilbert complex* (W, d)*, then*

$$\|v\|_V \leq c_P \|\mathrm{d}^k v\|, \quad \forall v \in \mathfrak{Z}^{k\perp}.$$

We close this subsection by defining the dual complex of a Hilbert complex, and recalling how the Hodge decomposition can be interpreted in terms of this complex.

Definition 2.8. Given a Hilbert complex (W, d), the *dual complex* (W^*, d^*) consists of the spaces $W_k^* = W^k$, and adjoint operators $\mathrm{d}_k^* = \left(\mathrm{d}^{k-1}\right)^* : V_k^* \subset W_k^* \to V_{k-1}^* \subset W_{k-1}^*$.

$$\cdots \longleftarrow V_{k-1}^* \xleftarrow{\mathrm{d}_k^*} V_k^* \xleftarrow{\mathrm{d}_{k+1}^*} V_{k+1}^* \longleftarrow \cdots$$

Remark 2. Since the arrows in the dual complex point in the opposite direction, this is a Hilbert chain complex rather than a cochain complex. (The chain property $\mathrm{d}_k^* \circ \mathrm{d}_{k+1}^* = 0$ follows immediately from the cochain property $\mathrm{d}^k \circ \mathrm{d}^{k-1} = 0$.) Accordingly, we can define the *k-cycles* $\mathfrak{Z}_k^* = \ker \mathrm{d}_k^* = \mathfrak{B}^{k\perp_W}$ and *k-boundaries* $\mathfrak{B}_k^* = \mathrm{d}_{k+1}^* V_k^*$. The kth harmonic space can then be rewritten as $\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{Z}_k^*$; we also have $\mathfrak{Z}^k = \mathfrak{B}_k^{*\perp_W}$, and thus $\mathfrak{Z}^{k\perp_W} = \overline{\mathfrak{B}_k^*}$. Therefore, the weak Hodge decomposition can be written as

$$W^k = \overline{\mathfrak{B}^k} \oplus \mathfrak{H}^k \oplus \overline{\mathfrak{B}_k^*},$$

and in particular, for a closed Hilbert complex, the strong Hodge decomposition now becomes

$$W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{B}_k^*.$$

2.3. **The abstract Hodge Laplacian and mixed variational problem.** The *abstract Hodge Laplacian* is the operator $L = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$, which is an unbounded operator $W^k \to W^k$ with domain

$$D_L = \left\{ u \in V^k \cap V_k^* \;\middle|\; \mathrm{d}u \in V_{k+1}^*,\ \mathrm{d}^* u \in V^{k-1} \right\}.$$

This is a generalization of the Hodge Laplacian for differential forms, which itself is a generalization of the usual scalar and vector Laplacian operators on domains in $\mathbb{R}^n$ (as well as of the Laplace–Beltrami operator on Riemannian manifolds).

If $u \in D_L$ solves $Lu = f$, then it satisfies the variational principle

$$\langle \mathrm{d}u, \mathrm{d}v \rangle + \langle \mathrm{d}^* u, \mathrm{d}^* v \rangle = \langle f, v \rangle, \quad \forall v \in V^k \cap V_k^*.$$

However, as noted by Arnold, Falk, and Winther [4], there are some difficulties in using this variational principle for a finite element approximation. First, it may be difficult to construct finite elements for the space $V^k \cap V_k^*$. A second concern is the well-posedness of the problem. If we take any harmonic test function $v \in \mathfrak{H}^k$, then the left-hand side vanishes, so $\langle f, v \rangle = 0$; hence, a solution only exists if $f \perp \mathfrak{H}^k$. Furthermore, for any $q \in \mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{Z}_k^*$, we have $\mathrm{d}q = 0$ and $\mathrm{d}^* q = 0$; therefore, if u is a solution, then so is $u + q$.

To avoid these existence and uniqueness issues, one instead defines the following mixed variational problem: Find $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ satisfying

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, \mathrm{d}\tau \rangle &= 0, && \forall \tau \in V^{k-1}, \\ \langle \mathrm{d}\sigma, v \rangle + \langle \mathrm{d}u, \mathrm{d}v \rangle + \langle p, v \rangle &= \langle f, v \rangle, && \forall v \in V^k, \\ \langle u, q \rangle &= 0, && \forall q \in \mathfrak{H}^k. \end{aligned} \tag{1}$$

Here, the first equation implies that $\sigma = \mathrm{d}^* u$, which weakly enforces the condition $u \in V^k \cap V_k^*$. Next, the second equation incorporates the additional term $\langle p, v \rangle$, which allows for solutions to exist even when $f \not\perp \mathfrak{H}^k$. Finally, the third equation fixes the issue of non-uniqueness by requiring $u \perp \mathfrak{H}^k$. The following result establishes the well-posedness of the problem (1).

Theorem 2.9 (Arnold, Falk, and Winther [4], Theorem 3.1)**.** *Let (W, d) be a closed Hilbert complex with domain complex (V, d). The mixed formulation of the abstract Hodge Laplacian is well-posed. That is, for any $f \in W^k$, there exists a unique $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ satisfying (1). Moreover,*

$$\|\sigma\|_V + \|u\|_V + \|p\| \leq c \|f\|,$$

where c is a constant depending only on the Poincaré constant c_P in Lemma 2.6.

To prove this, they observe that (1) can be rewritten as a standard variational problem—i.e., one having the form $B(x, y) = F(y)$—on the space $V^{k-1} \times V^k \times \mathfrak{H}^k$, by defining the bilinear form

$$B(\sigma, u, p; \tau, v, q) = \langle \sigma, \tau \rangle - \langle u, \mathrm{d}\tau \rangle + \langle \mathrm{d}\sigma, v \rangle + \langle \mathrm{d}u, \mathrm{d}v \rangle + \langle p, v \rangle - \langle u, q \rangle$$

and the functional $F(\tau, v, q) = \langle f, v \rangle$. The well-posedness of the mixed problem then follows by establishing the inf-sup condition for the bilinear form $B(\cdot, \cdot)$ [4, Theorem 3.2], which shows that it defines a linear homeomorphism. This well-posedness result implies the existence of a bounded solution operator $K \colon W^k \to W^k$ defined by $Kf = u$.

2.4. **Approximation by a subcomplex.** In order to obtain approximate numerical solutions to the mixed variational problem (1), Arnold, Falk, and Winther [4] suppose that one is given a (finite-dimensional) subcomplex $V_h \subset V$ of the domain complex: that is, $V_h^k \subset V^k$ is a Hilbert subspace for each k, and the inclusion mapping $i_h \colon V_h \hookrightarrow V$ is a morphism of Hilbert complexes. By analogy with the Galerkin method, one can then consider the mixed variational problem on the subcomplex: Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ satisfying

$$\begin{aligned} \langle \sigma_h, \tau \rangle - \langle u_h, \mathrm{d}\tau \rangle &= 0, && \forall \tau \in V_h^{k-1}, \\ \langle \mathrm{d}\sigma_h, v \rangle + \langle \mathrm{d}u_h, \mathrm{d}v \rangle + \langle p_h, v \rangle &= \langle f, v \rangle, && \forall v \in V_h^k, \\ \langle u_h, q \rangle &= 0, && \forall q \in \mathfrak{H}_h^k. \end{aligned} \tag{2}$$

For the error analysis of this method, one more crucial assumption must be made: that there exists some Hilbert complex "projection" $\pi_h \colon V \to V_h$. We put "projection" in quotes because this need not be the actual orthogonal projection i_h^* with respect to the inner product; indeed, that projection is not generally a morphism of Hilbert complexes, since it may not commute with the differentials. However, the map π_h is V-bounded, surjective, and idempotent. It follows, then, that although it does not satisfy the optimality property of the orthogonal projection, it does still satisfy a *quasi-optimality* property, since

$$\lVert u - \pi_h u \rVert_V = \inf_{v \in V_h} \lVert (I - \pi_h)(u - v) \rVert_V \leq \lVert I - \pi_h \rVert \inf_{v \in V_h} \lVert u - v \rVert_V ,$$

where the first step follows from the idempotence of π_h, i.e., $(I - \pi_h) v = 0$ for all $v \in V_h$. With this framework in place, the following error estimate can be established.

Theorem 2.10 (Arnold, Falk, and Winther [4], Theorem 3.9)**.** *Let* (V_h, d) *be a family of subcomplexes of the domain complex* (V, d) *of a closed Hilbert complex, parametrized by* h *and admitting uniformly* V*-bounded cochain projections, and let* $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ *be the solution of* (1) *and* $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ *the solution of problem* (2)*. Then*

$$\begin{aligned} &\lVert \sigma - \sigma_h \rVert_V + \lVert u - u_h \rVert_V + \lVert p - p_h \rVert \\ &\leq C \bigl(\inf_{\tau \in V_h^{k-1}} \lVert \sigma - \tau \rVert_V + \inf_{v \in V_h^k} \lVert u - v \rVert_V + \inf_{q \in V_h^k} \lVert p - q \rVert_V + \mu \inf_{v \in V_h^k} \lVert P_{\mathfrak{B}} u - v \rVert_V \bigr), \end{aligned}$$

where $\mu = \mu_h^k = \sup_{\substack{r \in \mathfrak{H}^k \\ \lVert r \rVert = 1}} \bigl\lVert \bigl(I - \pi_h^k \bigr) r \bigr\rVert$.

Therefore, if V_h is pointwise approximating, in the sense that $\inf_{v \in V_h} \lVert u - v \rVert \to 0$ as $h \to 0$ for every $u \in V$, then the numerical solution converges to the exact solution.

2.5. **Improved error estimates.** Finally, it can be shown that one can establish improved estimates in the W-norm, subject to a "compactness property." The Hilbert complex (W, d) is said to have the *compactness property* if $V^k \cap V_k^*$ is a dense subset of W^k, and if the inclusion $\mathcal{I} \colon V^k \cap V_k^* \hookrightarrow W^k$ is compact. Furthermore, assume that the family of projections π_h is uniformly W-bounded (rather than merely V-bounded) with respect to h. These properties hold for many important examples—notably the L^2-de Rham complex of differential forms—and allows for an abstract generalization of duality-based L^2 estimates (i.e., the Aubin–Nitsche trick) to the mixed variational problem.

The compactness of the inclusion implies that K is also compact, so one may define the coefficients

$$\begin{gathered} \delta = \delta_h^k = \lVert (I - \pi_h) K \rVert_{\mathcal{L}(W^k, W^k)} , \qquad \mu = \mu_h^k = \lVert (I - \pi_h) P_{\mathfrak{H}} \rVert_{\mathcal{L}(W^k, W^k)} , \\ \eta = \eta_h^k = \max_{j = 0,1} \Bigl\{ \lVert (I - \pi_h) \mathrm{d} K \rVert_{\mathcal{L}(W^{k-j}, W^{k-j+1})} , \lVert (I - \pi_h) \mathrm{d}^* K \rVert_{\mathcal{L}(W^{k+j}, W^{k+j-1})} \Bigr\} , \end{gathered}$$

each of which vanishes in the limit as $h \to 0$. Next, let us denote best approximation in the W-norm by

$$E(w) = \inf_{v \in V_h^k} \lVert w - v \rVert , \quad w \in W^k .$$

Then the improved estimates are stated in the following theorem.

Theorem 2.11 (Arnold, Falk, and Winther [4], Theorem 3.11)**.** *Let* (V, d) *be the domain complex of a closed Hilbert complex* (W, d) *satisfying the compactness property, and let* (V_h, d) *be a family of subcomplexes parametrized by* h *and admitting uniformly* W*-bounded cochain projections. Let* $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ *be the* *solution of* (1) *and* $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ *the solution of problem* (2). *Then for some constant* C *independent of* h *and* (σ, u, p), *we have*

$$\begin{aligned}
\lVert \mathrm{d}(\sigma - \sigma_h) \rVert &\le C E(\mathrm{d}\sigma), \\
\lVert \sigma - \sigma_h \rVert &\le C \left[E(\sigma) + \eta E(\mathrm{d}\sigma) \right], \\
\lVert p - p_h \rVert &\le C \left[E(p) + \mu E(\mathrm{d}\sigma) \right], \\
\lVert \mathrm{d}(u - u_h) \rVert &\le C \left(E(\mathrm{d}u) + \eta \left[E(\mathrm{d}\sigma) + E(p) \right] \right), \\
\lVert u - u_h \rVert &\le C \Big(E(u) + \eta \left[E(\mathrm{d}u) + E(\sigma) \right] \\
&\qquad + \left(\eta^2 + \delta \right) \left[E(\mathrm{d}\sigma) + E(p) \right] + \mu E\left(P_{\mathfrak{B}} u \right) \Big).
\end{aligned}$$

For typical applications to the de Rham complex, V_h^k consists of piecewise polynomials defined on a mesh. In this case, the order of these coefficients is given by $\eta = O(h)$, $\delta = O\left(h^{\min(2, r+1)}\right)$, and $\mu = O\left(h^{r+1}\right)$, where r is the largest degree of complete polynomials in V_h^k (Arnold, Falk, and Winther [4, p. 312]).

2.6. **Variational crimes.** More generally, suppose that the discrete complex V_h is not necessarily a subcomplex of V, but that we merely have a W-bounded inclusion map $i_h \colon V_h \hookrightarrow V$, which is a morphism of Hilbert complexes. Furthermore, given the V-bounded projection map $\pi_h \colon V \to V_h$, we require that $\pi_h^k \circ i_h^k = \mathrm{id}_{V_h^k}$ for each k (which corresponds to the idempotence of π_h when i_h is simply the inclusion of a subcomplex $V_h \subset V$). When i_h is unitary—that is, when the discrete inner product satisfies $\langle u_h, v_h \rangle_h = \langle i_h u_h, i_h v_h \rangle$ for all $u_h, v_h \in V_h^k$—then this is precisely equivalent to considering the subcomplex $i_h V_h \subset V$. However, if i_h is not necessarily unitary, we have a generalized version of the discrete variational problem (2), stated as follows: Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ satisfying

$$\begin{aligned}
\langle \sigma_h, \tau_h \rangle_h - \langle u_h, \mathrm{d}_h \tau_h \rangle_h &= 0, && \forall \tau_h \in V_h^{k-1}, \\
\langle \mathrm{d}_h \sigma_h, v_h \rangle_h + \langle \mathrm{d}_h u_h, \mathrm{d}_h v_h \rangle_h + \langle p_h, v_h \rangle_h &= \langle f_h, v_h \rangle_h, && \forall v_h \in V_h^k, \\
\langle u_h, q_h \rangle_h &= 0, && \forall q_h \in \mathfrak{H}_h^k.
\end{aligned} \tag{3}$$

The additional error in this generalized discretization, relative to the problem on the subcomplex $i_h V_h \subset V$, arises from two particular variational crimes: one resulting from the failure of i_h to be unitary, and another resulting from the difference between f_h and $i_h^* f$.

In Holst and Stern [24], we analyze this additional error by introducing a modified problem on V_h, which is equivalent to the subcomplex problem on $i_h V_h \subset V$. Define $J_h = i_h^* i_h$, so that for any $u_h, v_h \in W_h$, we have $\langle i_h u_h, i_h v_h \rangle = \langle i_h^* i_h u_h, v_h \rangle_h = \langle J_h u_h, v_h \rangle_h$. (The norm $\lVert I - J_h \rVert$, therefore, quantifies the failure of i_h to be unitary.) This defines a modified inner product on W_h^k, leading to a modified Hodge decomposition $W_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^{\prime k} \oplus \mathfrak{Z}_h^{k \perp' W}$, where

$$\mathfrak{H}_h^{\prime k} = \left\{ z \in \mathfrak{Z}_h^k \;\middle|\; i_h z \perp i_h \mathfrak{B}_h^k \right\}, \qquad \mathfrak{Z}_h^{k \perp' W} = \left\{ v \in W_h^k \;\middle|\; i_h v \perp i_h \mathfrak{Z}_h^k \right\}.$$

Then the subcomplex problem is equivalent to the following mixed problem: Find $(\sigma_h', u_h', p_h') \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^{\prime k}$ satisfying

$$\begin{aligned}
\langle J_h \sigma_h', \tau_h \rangle_h - \langle J_h u_h', \mathrm{d}_h \tau_h \rangle_h &= 0, && \forall \tau_h \in V_h^{k-1}, \\
\langle J_h \mathrm{d}_h \sigma_h', v_h \rangle_h + \langle J_h \mathrm{d}_h u_h', \mathrm{d}_h v_h \rangle_h + \langle J_h p_h', v_h \rangle_h &= \langle i_h^* f, v_h \rangle_h, && \forall v_h \in V_h^k, \\
\langle J_h u_h', q_h' \rangle_h &= 0, && \forall q_h' \in \mathfrak{H}_h^{\prime k}.
\end{aligned} \tag{4}$$

The additional error, between the generalized problem (3) and the subcomplex problem (4), is estimated in the following theorem.

Theorem 2.12 (Holst and Stern [24], Theorem 3.9). *Suppose that* $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ *is a solution to* (3) *and* $(\sigma'_h, u'_h, p'_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}'^k_h$ *is a solution to* (4). *Then*

$$\lVert \sigma_h - \sigma'_h \rVert_{V_h} + \lVert u_h - u'_h \rVert_{V_h} + \lVert p_h - p'_h \rVert_h \leq C \left(\lVert f_h - i_h^* f \rVert_h + \lVert I - J_h \rVert \, \lVert f \rVert \right).$$

Using the triangle inequality, together with the previously stated result of Arnold, Falk, and Winther (Theorem 2.10) for the subcomplex problem, we immediately get the following corollary.

Corollary 2.13 (Holst and Stern [24], Corollary 3.10). *If* $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ *is a solution to* (1) *and* $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ *is a solution to* (3), *then*

$$\begin{aligned} &\lVert \sigma - i_h \sigma_h \rVert_V + \lVert u - i_h u_h \rVert_V + \lVert p - i_h p_h \rVert \\ &\leq C \Big(\inf_{\tau \in i_h V_h^{k-1}} \lVert \sigma - \tau \rVert_V + \inf_{v \in i_h V_h^k} \lVert u - v \rVert_V + \inf_{q \in i_h V_h^k} \lVert p - q \rVert_V + \mu \inf_{v \in i_h V_h^k} \lVert P_{\mathfrak{B}} u - v \rVert_V \\ &\qquad + \lVert f_h - i_h^* f \rVert_h + \lVert I - J_h \rVert \, \lVert f \rVert \Big), \end{aligned}$$

where μ *is defined as in Theorem 2.10.*

This raises the question of how to choose $f_h \in W_h$ such that $f_h \to i_h^* f$ as $h \to 0$. While $f_h = i_h^* f$ would be the ideal choice, of course, it may be difficult to compute the inner product on W, and hence to compute the adjoint i_h^*. The following result shows that, if $\Pi_h \colon W^k \to W_h^k$ is any bounded linear projection (i.e., satisfying $\Pi_h \circ i_h^k = \mathrm{id}_{W_h^k}$), then choosing $f_h = \Pi_h f$ is sufficient to control this term.

Theorem 2.14 (Holst and Stern [24], Theorem 3.11). *If* $\Pi_h \colon W^k \to W_h^k$ *is a family of linear projections, bounded uniformly with respect to* h*, then we have the inequality*

$$\lVert \Pi_h f - i_h^* f \rVert_h \leq C \Big(\lVert I - J_h \rVert \, \lVert f \rVert + \inf_{\phi \in i_h W_h^k} \lVert f - \phi \rVert \Big).$$

Thus, if the family of discrete complexes satisfies the "well-approximating" condition, and if $\lVert I - J_h \rVert \to 0$ as $h \to 0$, then it follows that the generalized discrete solution converges to the continuous solution.

3. Semilinear mixed problems

3.1. An alternative approach to the linear problem.

In this subsection, we introduce a slightly modified approach to the linear problem, which will be more useful in the nonlinear analysis to follow.

Consider the linear operator $\mathbf{L} = L \oplus P_{\mathfrak{H}} \colon D_L \to W^k$. Given any $\mathbf{u} \in D_L$, we can orthogonally decompose $\mathbf{u} = u + p$, where $p = P_{\mathfrak{H}} \mathbf{u}$ and $u = \mathbf{u} - p$. Therefore,

$$\mathbf{L}\mathbf{u} = L\mathbf{u} + P_{\mathfrak{H}} \mathbf{u} = Lu + p,$$

so given some $f \in W^k$, solving $Lu + p = f$ is equivalent to solving $\mathbf{L}\mathbf{u} = f$. Furthermore, if we define the solution operator $\mathbf{K} = K \oplus P_{\mathfrak{H}}$, it follows that

$$\mathbf{K} f = K f + P_{\mathfrak{H}} f = u + p = \mathbf{u},$$

so $\mathbf{K}$ is in fact the inverse of $\mathbf{L}$. Thus, $\mathbf{L}$ and $\mathbf{K}$ establish a bijection between D_L and W^k. Effectively, by adding $P_{\mathfrak{H}}$ to each of the operators L and K, we have managed to remove their kernel $\mathfrak{H}^k$.

This approach also sheds new light on the well-posedness of the linear problem. If $\mathbf{u}$ is a solution to $\mathbf{L}\mathbf{u} = f$, then it satisfies the variational problem: Find $\mathbf{u} \in V^k \cap V_k^*$ such that

$$\langle \mathrm{d}^*\mathbf{u}, \mathrm{d}^* v \rangle + \langle \mathrm{d}\mathbf{u}, \mathrm{d}v \rangle + \langle P_{\mathfrak{H}}\mathbf{u}, P_{\mathfrak{H}} v \rangle = \langle f, v \rangle , \quad \forall v \in V^k \cap V_k^*. \tag{5}$$

In fact, the left-hand side is precisely the inner product $\langle \mathbf{u}, v \rangle_{V \cap V^*}$, which is equivalent to the usual intersection inner product obtained by adding the inner products for V and V^* (Arnold, Falk, and Winther [4, p. 312]). Hence, by the Riesz representation theorem, a unique solution $\mathbf{u} = \mathbf{K} f$ exists, and moreover $\mathbf{K}$ is bounded. In particular, this variational formulation also illustrates that $\mathbf{K}$ is the adjoint to the bounded inclusion $\mathcal{I} \colon V^k \cap V_k^* \hookrightarrow W^k$, with respect to this $\langle \cdot, \cdot \rangle_{V \cap V^*}$ inner product, and thus $\mathbf{K}$ must be bounded as well.

Remark 3. While the solutions to the two variational problems (1) and (5) are equivalent, the mixed formulation is still preferable for implementing finite element methods, since one may not have efficient finite elements for the space $V^k \cap V_k^*$. We emphasize that this alternative approach is introduced primarily to make the analysis of semilinear problems more convenient.

3.2. **Semilinear problems and the abstract Hammerstein equation.** Given some $f \in W^k$, we are interested in the semilinear problem of finding $\mathbf{u}$, such that

$$\mathbf{L}\mathbf{u} + F\mathbf{u} = f, \tag{6}$$

where $F \colon V^k \to W^k$ is some nonlinear operator. Extending the argument from the linear case, it follows that this operator equation is equivalent to the mixed variational problem: Find $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ satisfying

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, \mathrm{d}\tau \rangle &= 0, & \forall \tau \in V^{k-1} \\ \langle \mathrm{d}\sigma, v \rangle + \langle \mathrm{d}u, \mathrm{d}v \rangle + \langle p, v \rangle + \langle F(u+p), v \rangle &= \langle f, v \rangle , & \forall v \in V^k, \\ \langle u, q \rangle &= 0, & \forall q \in \mathfrak{H}^k. \end{aligned} \tag{7}$$

In the special case where $F = 0$, this simply reduces to the linear problem.

Using the solution operator $\mathbf{K}$, the equation (6) is also equivalent to

$$\mathbf{u} + \mathbf{K}F\mathbf{u} = \mathbf{K}f. \tag{8}$$

Equations having this general form are called *abstract Hammerstein equations*, and are of particular interest in nonlinear functional analysis (cf. Zeidler [30]). This formulation, which notably appeared in the seminal papers of Amann [1, 2] and Browder and Gupta [8], generalizes certain nonlinear integral equations, called Hammerstein integral equations. (In the context of integral equations, the operator $\mathbf{K}$ corresponds to the kernel operator, or Green's operator.)

3.3. **Well-posedness of the semilinear problem.** Before we establish the well-posedness of the abstract Hammerstein equation (8), it is necessary to define some special properties that a nonlinear operator may have.

Definition 3.1. The operator $A \colon W^k \to W^k$ is said to be *monotone* if, for all $u, v \in W^k$, it satisfies $\langle Au - Av, u - v \rangle \geq 0$. It is called *strictly monotone* if $\langle Au - Av, u - v \rangle > 0$ whenever $u \neq v$, and *strongly monotone* if there exists a constant $c > 0$ such that $\langle Au - Av, u - v \rangle \geq c \left\lVert u - v \right\rVert^2$.

Definition 3.2. The operator $A\colon W^k \to W^k$ is said to be *hemicontinuous* if the real function $t \mapsto \langle A(u+tv), w\rangle$ is continuous on $[0,1]$ for all $u, v, w \in W^k$.

Theorem 3.3. *If F is monotone and hemicontinuous, then the semilinear problem* (6) *has a unique solution. Moreover, the problem is well-posed: given two functionals f and f', the respective solutions $\mathbf{u}$ and $\mathbf{u}'$ satisfy the Lipschitz continuity estimate $\|\mathbf{u}-\mathbf{u}'\|_{V\cap V^*} \le \|\mathbf{K}\| \, \|f-f'\|$.*

The existence/uniqueness portion of the proof is an adaptation of a standard argument for Hammerstein equations, when the kernel operator is symmetric and monotone on some real, separable Hilbert space (cf. Zeidler [30, p. 618]).

Proof. Let us define the operator $A = I + \mathbf{K}F$ on $V^k \cap V_k^*$, so that the abstract Hammerstein equation (8) can be written as $A\mathbf{u} = \mathbf{K}f$. Since F is hemicontinuous, it follows that A is also hemicontinuous. Moreover, A is strongly monotone with constant $c = 1$, since for any $\mathbf{u}, \mathbf{u}' \in V^k \cap V_k^*$, we have

$$\begin{aligned} \langle A\mathbf{u} - A\mathbf{u}', \mathbf{u}-\mathbf{u}'\rangle_{V\cap V^*} &= \|\mathbf{u}-\mathbf{u}'\|_{V\cap V^*}^2 + \langle \mathbf{K}(F\mathbf{u} - F\mathbf{u}'), \mathbf{u}-\mathbf{u}'\rangle_{V\cap V^*} \\ &= \|\mathbf{u}-\mathbf{u}'\|_{V\cap V^*}^2 + \langle F\mathbf{u} - F\mathbf{u}', \mathbf{u}-\mathbf{u}'\rangle \\ &\ge \|\mathbf{u}-\mathbf{u}'\|_{V\cap V^*}^2 , \end{aligned}$$

where the last line follows from the monotonicity of F. Therefore, since A is hemicontinuous and strongly monotone, the Browder–Minty theorem [7, 25] implies that it has a Lipschitz continuous inverse A^{-1} with Lipschitz constant $c^{-1} = 1$. Hence, there exist unique solutions $\mathbf{u} = A^{-1}\mathbf{K}f$ and $\mathbf{u}' = A^{-1}\mathbf{K}f'$. Finally, by the fact that A^{-1} is nonexpansive, these solutions satisfy

$$\|\mathbf{u}-\mathbf{u}'\|_{V\cap V^*} \le \|\mathbf{K}f - \mathbf{K}f'\|_{V\cap V^*} \le \|\mathbf{K}\| \, \|f-f'\| ,$$

which completes the proof. ☐

3.4. **Solution estimate for the mixed formulation.** Now that we have established the well-posedness of the semilinear problem (6), we can use the *linear* solution theory, as developed by Arnold, Falk, and Winther [4], to develop a similar estimate for the mixed formulation. This requires placing slightly stronger conditions on the nonlinear operator F. In particular, we require F to be Lipschitz continuous with respect to the V-norm: that is, there exists a constant C such that

$$\|F\mathbf{u} - F\mathbf{u}'\| \le C \, \|\mathbf{u}-\mathbf{u}'\|_V ,$$

for all $\mathbf{u}, \mathbf{u}' \in V^k$. (Later, in Section 4.5, we will see how this condition can be relaxed in case F is only locally Lipschitz.)

Theorem 3.4. *If F is monotone and Lipschitz continuous with respect to the V-norm, then the mixed semilinear problem* (7) *has a unique solution (σ, u, p). Moreover, the problem is well-posed: given two functionals f and f', the respective solutions (σ, u, p) and (σ', u', p') satisfy the Lipschitz continuity estimate*

$$\|\sigma-\sigma'\|_V + \|u-u'\|_V + \|p-p'\| \le C \, \|f-f'\| ,$$

where the constant C depends only on the Poincaré constant c_P and on the Lipschitz constant of F.

Proof. If $\mathbf{u}$ is a solution of the semilinear problem $\mathbf{L}\mathbf{u} + F\mathbf{u} = f$, then it is also a solution of the linear problem $\mathbf{L}\mathbf{u} = g$, where $g = f - F\mathbf{u}$. Therefore, $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ is the unique solution of the mixed linear problem with functional g, and hence of the mixed semilinear problem (7).

Now, suppose that $\mathbf{u}'$ is the solution to $\mathbf{L}\mathbf{u}' + F\mathbf{u}' = f'$, and hence to the linear problem $\mathbf{L}\mathbf{u}' = g' = f' - F\mathbf{u}'$. Define $\overline{\mathbf{u}} = \mathbf{u} - \mathbf{u}'$ and $\overline{g} = g - g'$; subtracting the two linear equations $\mathbf{L}\mathbf{u} = g$ and $\mathbf{L}\mathbf{u}' = g'$, it follows that $\mathbf{L}\overline{\mathbf{u}} = \overline{g}$. Therefore, $(\overline{\sigma}, \overline{u}, \overline{p}) = (\sigma - \sigma', u - u', p - p')$ satisfies the mixed linear problem with functional $\overline{g}$, so by the well-posedness of the mixed linear problem, we have

$$\|\overline{\sigma}\|_V + \|\overline{u}\|_V + \|\overline{p}\| \leq c\,\|\overline{g}\|,$$

where c depends only on the Poincaré constant c_P. Next, the right-hand side can be estimated by

$$\begin{aligned}\|\overline{g}\| &\leq \|f - f'\| + \|F\mathbf{u} - F\mathbf{u}'\| \\ &\leq \|f - f'\| + C\,\|\mathbf{u} - \mathbf{u}'\|_V \leq \|f - f'\| + C\,\|\mathbf{u} - \mathbf{u}'\|_{V \cap V^*},\end{aligned}$$

using the Lipschitz property of F. Finally, applying the previously-obtained estimate $\|\mathbf{u} - \mathbf{u}'\|_{V \cap V^*} \leq \|\mathbf{K}\|\,\|f - f'\|$, we get $\|\overline{g}\| \leq C\,\|f - f'\|$, so finally

$$\|\sigma - \sigma'\|_V + \|u - u'\|_V + \|p - p'\| \leq C\,\|f - f'\|,$$

which completes the proof. □

Remark 4. Note that, in the linear case where $F = 0$, we can take $f' = 0$ so that $(\sigma', u', p') = 0$. Then, since $g = f$ and $g' = f' = 0$, we simply recover the usual linear estimate $\|\sigma\|_V + \|u\|_V + \|p\| \leq c\,\|f\|$.

4. Approximation theory and numerical analysis

4.1. **The discrete semilinear problem.** To set up the discrete semilinear problem, and develop the subsequent convergence results, we begin by assuming the same conditions as in the linear case. Namely, suppose that $V_h \subset V$ is a Hilbert subcomplex, equipped with a bounded cochain projection $\pi_h \colon V \to V_h$. Let $K_h \colon W_h^k \to W_h^k$ be the discrete solution operator for the linear problem, taking $P_h f \mapsto u_h$. As with the continuous problem, we define a new solution operator $\mathbf{K}_h = K_h \oplus P_{\mathfrak{H}_h}$ and consider the discrete Hammerstein equation

$$\mathbf{u}_h + \mathbf{K}_h P_h F \mathbf{u}_h = \mathbf{K}_h P_h f.$$

Note that this is *not* simply the Galerkin problem for the original Hammerstein operator equation (8), since $\mathbf{K}_h$ is not just a projection of $\mathbf{K}$ onto the discrete space; in particular, we generally have $\mathfrak{H}_h^k \not\subset \mathfrak{H}^k$.

This is precisely the abstract Hammerstein equation on the discrete Hilbert complex V_h, in the sense of the previous section. Therefore, there exists a unique solution $\mathbf{u}_h$, and the discrete solution operator $P_h f \mapsto \mathbf{u}_h$, $P_h f' \mapsto \mathbf{u}_h'$, satisfies the Lipschitz condition

$$\|\mathbf{u}_h - \mathbf{u}_h'\|_{V_h \cap V_h^*} \leq \|\mathbf{K}_h\|\,\|P_h\,(f - f')\| \leq \|\mathbf{K}_h\|\,\|f - f'\|.$$

Equivalently, this gives a solution to the discrete mixed variational problem: Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ satisfying

$$\begin{aligned} \langle \sigma_h, \tau \rangle - \langle u_h, \mathrm{d}\tau \rangle &= 0, && \forall \tau \in V_h^{k-1}, \\ \langle \mathrm{d}\sigma_h, v \rangle + \langle \mathrm{d}u_h, \mathrm{d}v \rangle + \langle p_h, v \rangle + \langle F(u_h + p_h), v \rangle &= \langle f, v \rangle, && \forall v \in V_h^k, \\ \langle u_h, q \rangle &= 0, && \forall q \in \mathfrak{H}_h^k. \end{aligned} \tag{9}$$

If F is Lipschitz, then we also obtain an estimate for the mixed solution,

$$\|\sigma_h - \sigma_h'\|_V + \|u_h - u_h'\|_V + \|p_h - p_h'\| \le C_h \|f - f'\|.$$

Finally, we remark that when V_h is a family of subcomplexes parametrized by h, and the projections $\pi_h \colon V \to V_h$ are bounded uniformly with respect to h, then the constants in these estimates may also be bounded independently of h.

4.2. **Convergence of the discrete solution.** We now estimate the error in approximating the solution of the mixed semilinear problem (7) by that for the discrete problem (9). Despite the introduction of nonlinearity, we obtain the same quasi-optimal estimate as in Theorem 2.10 for the linear problem.

Theorem 4.1. *Let (V_h, d) be a family of subcomplexes of the domain complex (V, d) of a closed Hilbert complex, parametrized by h and admitting uniformly V-bounded cochain projections, and let $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ be the solution of (7) and $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ the solution of problem (9). Then, assuming the operator F is Lipschitz with respect to the V-norm, we have the estimate*

$$\begin{aligned} &\|\sigma - \sigma_h\|_V + \|u - u_h\|_V + \|p - p_h\| \\ &\le C\Big(\inf_{\tau \in V_h^{k-1}} \|\sigma - \tau\|_V + \inf_{v \in V_h^k} \|u - v\|_V + \inf_{q \in V_h^k} \|p - q\|_V + \mu \inf_{v \in V_h^k} \|P_{\mathfrak{B}} u - v\|_V \Big), \end{aligned}$$

where μ is defined as in Theorem 2.10, and where the constant C depends only on the Poincaré constant c_P and the Lipschitz constant of F.

Proof. Recall that, since (σ, u, p) solves the semilinear problem for the functional f, it also solves the linear problem for the functional $g = f - F(u + p)$. Let $(\sigma_h', u_h', p_h') \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ be the solution to the corresponding discrete linear problem for g. By Theorem 2.10, this satisfies the error estimate

$$\begin{aligned} &\|\sigma - \sigma_h'\|_V + \|u - u_h'\|_V + \|p - p_h'\| \\ &\le C\Big(\inf_{\tau \in V_h^{k-1}} \|\sigma - \tau\|_V + \inf_{v \in V_h^k} \|u - v\|_V + \inf_{q \in V_h^k} \|p - q\|_V + \mu \inf_{v \in V_h^k} \|P_{\mathfrak{B}} u - v\|_V \Big). \end{aligned}$$

Next, observe that (σ_h', u_h', p_h') is also a solution of the discrete semilinear problem with functional $f' = f - F(u + p) + F(u_h' + p_h')$, since we can just add $F(u_h' + p_h')$ to both sides of the equation. However, since the discrete solution operator is Lipschitz, we have

$$\begin{aligned} \|\sigma_h - \sigma_h'\|_V + \|u_h - u_h'\|_V + \|p_h - p_h'\| &\le C \|f - f'\| \\ &= C \|F(u + p) - F(u_h' + p_h')\|. \end{aligned}$$

Furthermore, since F is also Lipschitz,

$$\|F(u + p) - F(u_h' + p_h')\| \le C \left(\|u - u_h'\|_V + \|p - p_h'\| \right),$$

which implies

$$\|\sigma_h - \sigma_h'\|_V + \|u_h - u_h'\|_V + \|p_h - p_h'\| \leq C\left(\|\sigma - \sigma_h'\|_V + \|u - u_h'\|_V + \|p - p_h'\|\right).$$

An application of the triangle inequality completes the proof. □

As in the linear case, this implies that if V_h is pointwise approximating in V as $h \to 0$, then $(\sigma_h, u_h, p_h) \to (\sigma, u, p)$. Moreover, the rate of convergence for this semilinear problem is the same as that for the linear problem.

4.3. **Improved estimates.** We now establish improved estimates for the semilinear problem, subject to the compactness property introduced in Section 2.5.

Theorem 4.2. *Let (V, d) be the domain complex of a closed Hilbert complex (W, d) satisfying the compactness property, and let (V_h, d) be a family of subcomplexes parametrized by h and admitting uniformly W-bounded cochain projections. Let $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ be the solution of (7) and $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ the solution of problem (9), and assume that the operator F is Lipschitz. Then for some constant C independent of h and (σ, u, p), we have*

$$\begin{aligned}
\|\mathrm{d}(\sigma - \sigma_h)\| &\leq C\big[E(\mathrm{d}\sigma) + E(u) + E(\mathrm{d}u) + E(p) \\
&\qquad + \eta E(\sigma) + \mu E(P_{\mathfrak{B}} u)\big] \\
\|\sigma - \sigma_h\| &\leq C\big[E(\sigma) + E(u) + E(\mathrm{d}u) + E(p) \\
&\qquad + (\eta + \delta + \mu) E(\mathrm{d}\sigma) + \mu E(P_{\mathfrak{B}} u)\big] \\
\|u - u_h\|_V + \|p - p_h\| &\leq C\big(E(u) + E(\mathrm{d}u) + E(p) \\
&\qquad + \eta\left[E(\sigma) + E(\mathrm{d}\sigma)\right] + (\delta + \mu) E(\mathrm{d}\sigma) + \mu E(P_{\mathfrak{B}} u)\big).
\end{aligned}$$

Proof. As before, let $(\sigma_h', u_h', p_h') \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ be the solution to the discrete linear problem with right-hand side functional $g = f - F(u + p)$. Then Theorem 2.11 gives the improved estimates

$$\begin{aligned}
\|\mathrm{d}(\sigma - \sigma_h')\| &\leq C E(\mathrm{d}\sigma), \\
\|\sigma - \sigma_h'\| &\leq C\left[E(\sigma) + \eta E(\mathrm{d}\sigma)\right], \\
\|p - p_h'\| &\leq C\left[E(p) + \mu E(\mathrm{d}\sigma)\right], \\
\|\mathrm{d}(u - u_h')\| &\leq C\left(E(\mathrm{d}u) + \eta\left[E(\mathrm{d}\sigma) + E(p)\right]\right), \\
\|u - u_h'\| &\leq C\big(E(u) + \eta\left[E(\mathrm{d}u) + E(\sigma)\right] \\
&\qquad + \left(\eta^2 + \delta\right)\left[E(\mathrm{d}\sigma) + E(p)\right] + \mu E(P_{\mathfrak{B}} u)\big).
\end{aligned}$$

However, in the proof of Theorem 4.1, we saw that each of the terms $\|\mathrm{d}(\sigma_h - \sigma_h')\|$, $\|\sigma_h - \sigma_h'\|$, and $\|u_h - u_h'\|_V + \|p_h - p_h'\|$ is controlled by

$$\begin{aligned}
\|\sigma_h - \sigma_h'\|_V &+ \|u_h - u_h'\|_V + \|p_h - p_h'\| \\
&\leq C\left(\|u - u_h'\|_V + \|p - p_h'\|\right) \\
&\leq C\big(E(u) + E(\mathrm{d}u) + E(p) \\
&\qquad + \eta\left[E(\sigma) + E(\mathrm{d}\sigma)\right] + (\delta + \mu) E(\mathrm{d}\sigma) + \mu E(P_{\mathfrak{B}} u)\big).
\end{aligned}$$

Applying the triangle inequality and eliminating higher-order terms, the result follows immediately. □

4.4. **Semilinear variational crimes.** As first discussed in Section 2.6, suppose now that V_h is not necessarily a subcomplex of V, and let $i_h \colon V_h \hookrightarrow V$ and $\pi_h \colon V \to V_h$ be the W-bounded inclusion and V-bounded projection morphisms, respectively, satisfying $\pi_h \circ i_h = \mathrm{id}_{V_h}$. Given a discrete functional $f_h \in W_h^k$ and a discrete nonlinear operator $F_h \colon V_h^k \to W_h^k$, we wish to approximate the continuous variational problem (7) by the discrete problem: Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ satisfying

$$
\begin{aligned}
\langle \sigma_h, \tau_h \rangle_h - \langle u_h, \mathrm{d}_h \tau_h \rangle_h &= 0, & \forall \tau_h \in V_h^{k-1}, \\
\langle \mathrm{d}_h \sigma_h, v_h \rangle_h + \langle \mathrm{d}_h u_h, \mathrm{d}_h v_h \rangle_h + \langle p_h, v_h \rangle_h & \\
+ \langle F_h(u_h + p_h), v_h \rangle_h &= \langle f_h, v_h \rangle_h, & \forall v_h \in V_h^k, \\
\langle u_h, q_h \rangle_h &= 0, & \forall q_h \in \mathfrak{H}_h^k.
\end{aligned}
\tag{10}
$$

For the following error estimate, we define the projection map $P_{V_h} \colon V \to V_h$ so that $i_h P_{V_h} v$ is the V-orthogonal projection of v onto the subcomplex $i_h V_h \subset V$.

Theorem 4.3. *Let $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ be the solution to (7) and $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ be the solution to (10). If F_h is Lipschitz, and its constant is uniformly bounded in h, then*

$$
\begin{aligned}
&\lVert \sigma - i_h \sigma_h \rVert_V + \lVert u - i_h u_h \rVert_V + \lVert p - i_h p_h \rVert \\
&\leq C \Big(\inf_{\tau \in i_h V_h^{k-1}} \lVert \sigma - \tau \rVert_V + \inf_{v \in i_h V_h^k} \lVert u - v \rVert_V + \inf_{q \in i_h V_h^k} \lVert p - q \rVert_V + \mu \inf_{v \in i_h V_h^k} \lVert P_{\mathfrak{B}} u - v \rVert_V \\
&\quad + \lVert i_h^* (f - F(u+p)) - (f_h - F_h P_{V_h}(u+p)) \rVert_h + \lVert I - J_h \rVert \lVert f - F(u+p) \rVert \Big),
\end{aligned}
$$

where μ is defined as in Theorem 2.10.

Proof. Suppose $(\sigma_h', u_h', p_h') \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ is the solution to the discrete linear problem with right-hand side functional $i_h^* g = i_h^* (f - F(u+p))$. Then, applying Corollary 2.13, we have

$$
\begin{aligned}
&\lVert \sigma - i_h \sigma_h' \rVert_V + \lVert u - i_h u_h' \rVert_V + \lVert p - i_h p_h' \rVert \\
&\leq C \Big(\inf_{\tau \in i_h V_h^{k-1}} \lVert \sigma - \tau \rVert_V + \inf_{v \in i_h V_h^k} \lVert u - v \rVert_V + \inf_{q \in i_h V_h^k} \lVert p - q \rVert_V + \mu \inf_{v \in i_h V_h^k} \lVert P_{\mathfrak{B}} u - v \rVert_V \\
&\qquad + \lVert I - J_h \rVert \lVert f - F(u+p) \rVert \Big).
\end{aligned}
$$

Next, observe that (σ_h', u_h', p_h') also solves the discrete semilinear problem with right-hand side functional $f_h' = i_h^* (f - F(u+p)) + F_h(u_h' + p_h')$. Therefore, since the discrete solution operator is Lipschitz, we obtain

$$
\begin{aligned}
&\lVert \sigma_h - \sigma_h' \rVert_{V_h} + \lVert u_h - u_h' \rVert_{V_h} + \lVert p_h - p_h' \rVert_h \\
&\qquad \leq C \lVert i_h^* (f - F(u+p)) - (f_h - F_h(u_h' + p_h')) \rVert_h \\
&\qquad \leq C \lVert i_h^* (f - F(u+p)) - (f_h - F_h P_{V_h}(u+p)) \rVert_h \\
&\qquad\quad + \lVert F_h P_{V_h}(u+p) - F_h(u_h' + p_h') \rVert_h .
\end{aligned}
$$

Applying the Lipschitz property of F_h to the last term of this expression,

$$
\begin{aligned}
\lVert F_h P_{V_h}(u+p) - F_h(u_h' + p_h') \rVert_h &\leq C \left(\lVert P_{V_h} u - u_h' \rVert_{V_h} + \lVert P_{V_h} p - p_h' \rVert_{V_h} \right) \\
&= C \left(\lVert P_{V_h}(u - i_h u_h') \rVert_{V_h} + \lVert P_{V_h}(p - i_h p_h') \rVert_{V_h} \right) \\
&\leq C \left(\lVert u - i_h u_h' \rVert_V + \lVert p - i_h p_h' \rVert \right),
\end{aligned}
$$

which we have already controlled. Hence, an application of the triangle inequality completes the proof. □

Clearly, the optimal choice for the functional f_h and the operator F_h would be

$$f_h = i_h^* f, \qquad F_h = i_h^* F i_h.$$

In this case, we would obtain

$$\begin{aligned}
&\left\| i_h^* \left(f - F\left(u+p\right) \right) - \left(f_h - F_h P_{V_h} \left(u+p\right) \right) \right\|_h \\
&\qquad = \left\| i_h^* \left(F\left(u+p\right) - F i_h P_{V_h} \left(u+p\right) \right) \right\|_h \\
&\qquad \le C \left\| \left(I - i_h P_{V_h} \right) \left(u+p\right) \right\|_V \\
&\qquad \le C \left(\inf_{v \in i_h V_h^k} \left\| u - v \right\|_V + \inf_{q \in i_h V_h^k} \left\| p - q \right\|_V \right),
\end{aligned}$$

which already appears elsewhere in the estimate. Hence, this choice of f_h and F_h allows the term $\left\| i_h^* \left(f - F\left(u+p\right) \right) - \left(f_h - F_h \left(P_h u + P_h p \right) \right) \right\|_h$ to be dropped.

However, as noted before, it may not be feasible to take $f_h = i_h^* f$ or $F_h = i_h^* F i_h$, since it is often difficult to compute the adjoint i_h^* to the inclusion. Instead, letting $\Pi_h \colon W^k \to W_h^k$ be any bounded linear projection, suppose we choose $f_h = \Pi_h f$ and $F_h = \Pi_h F i_h$, effectively approximating i_h^* by Π_h. As in the linear case, this choice will give us good convergence behavior, contributing an error that is again controlled by other terms in the error estimate.

Theorem 4.4. *Given a family of linear projections $\Pi_h \colon W^k \to W_h^k$, bounded uniformly with respect to h, suppose that $f_h = \Pi_h f$ and $F_h = \Pi_h F i_h$, where F is assumed to be Lipschitz. Then*

$$\begin{aligned}
\left\| i_h^* \left(f - F\left(u+p\right) \right) - \left(f_h - F_h P_{V_h} \left(u+p\right) \right) \right\|_h &\le C \big(\left\| I - J_h \right\| \left\| f - F\left(u+p\right) \right\| \\
+ \inf_{\phi \in i_h W_h^k} \left\| \left(f - F\left(u+p\right) \right) - \phi \right\| &+ \inf_{v \in i_h V_h^k} \left\| u - v \right\|_V + \inf_{q \in i_h V_h^k} \left\| p - q \right\|_V \big).
\end{aligned}$$

Proof. We begin by using the triangle inequality to write

$$\begin{aligned}
&\left\| i_h^* \left(f - F\left(u+p\right) \right) - \Pi_h \left(f - F i_h P_{V_h} \left(u+p\right) \right) \right\|_h \\
&\qquad \le \left\| \left(i_h^* - \Pi_h \right) \left(f - F\left(u+p\right) \right) \right\|_h + \left\| \Pi_h \left(F\left(u+p\right) - F i_h P_{V_h} \left(u+p\right) \right) \right\|_h .
\end{aligned}$$

For the first term, we can apply Theorem 2.14 to obtain

$$\begin{aligned}
&\left\| \left(i_h^* - \Pi_h \right) \left(f - F\left(u+p\right) \right) \right\|_h \\
&\qquad \le C \big(\left\| I - J_h \right\| \left\| f - F\left(u+p\right) \right\| + \inf_{\phi \in i_h W_h^k} \left\| \left(f - F\left(u+p\right) \right) - \phi \right\| \big).
\end{aligned}$$

For the remaining term, we have

$$\begin{aligned}
\left\| \Pi_h \left(F\left(u+p\right) - F i_h P_{V_h} \left(u+p\right) \right) \right\|_h &\le C \left\| F\left(u+p\right) - F i_h P_{V_h} \left(u+p\right) \right\| \\
&\le C \left\| \left(I - i_h P_{V_h} \right) \left(u+p\right) \right\|_V \\
&\le C \left(\inf_{v \in i_h V_h^k} \left\| u - v \right\|_V + \inf_{q \in i_h V_h^k} \left\| p - q \right\|_V \right),
\end{aligned}$$

which completes the proof. □

Hence, we again get convergence of the discrete solution to the continuous solution, as long as the discrete complex is well-approximating and $\left\| I - J_h \right\| \to 0$ as $h \to 0$.

4.5. **Remarks on relaxing the Lipschitz assumption.** Our *a priori* estimates for the mixed semilinear problem depended, crucially, on the assumption that the monotone operator F was not merely hemicontinuous but Lipschitz. In many problems of interest, however, F may be only *locally* Lipschitz: that is, given $\mathbf{u} \in V^k$, there exist constants $C, M > 0$ (possibly depending on $\mathbf{u}$) such that $\|F\mathbf{u} - F\mathbf{u}'\| \leq C \|\mathbf{u} - \mathbf{u}'\|_V$ whenever $\|\mathbf{u} - \mathbf{u}'\|_V \leq M$. What can we say about well-posedness and convergence when the Lipschitz condition is only local rather than global?

Since Theorem 3.3 requires only the hemicontinuity of F, we still know that the semilinear problem has a unique solution, and that it satisfies

$$\|\mathbf{u} - \mathbf{u}'\|_{V \cap V^*} \leq \|\mathbf{K}\| \|f - f'\| .$$

For the mixed problem, though, all we can show is that

$$\|\sigma - \sigma'\|_V + \|u - u'\|_V + \|p - p'\| \leq C \left(\|f - f'\| + \|F\mathbf{u} - F\mathbf{u}'\| \right),$$

at which point the proof of Theorem 3.4 requires the Lipschitz condition to continue. However, if F is locally Lipschitz at $\mathbf{u}$, then we can still proceed to obtain

$$\|\sigma - \sigma'\|_V + \|u - u'\|_V + \|p - p'\| \leq C \|f - f'\| ,$$

as long as $\|f - f'\|$ (and therefore $\|\mathbf{u} - \mathbf{u}'\|_V$) is sufficiently small. The same holds true for the well-posedness of the discrete mixed problem on V_h.

Now, let us observe how this affects the convergence of the discrete problem. In the proof of the *a priori* estimate, Theorem 4.1, we had

$$\|f - f'\| = \|F(u + p) - F(u_h' + p_h')\| ,$$

where (σ_h', u_h', p_h') is the solution to the discrete linear problem with right-hand side functional $g = f - F(u + p)$. If V_h is well-approximating in V, then Theorem 2.10 imples that, by taking h sufficiently small, we can get $\|f - f'\|$ to be as small as we want. Therefore, the error estimates hold as long as h is sufficiently small.

As an example of how these Lipschitz conditions arise, consider the following semilinear elliptic problem on a smooth, connected, open domain $\Omega \subset \mathbb{R}^n$: Find $u \in \mathring{H}^1(\Omega)$ such that

$$-\Delta u + u^m = f, \tag{11}$$

where $m \geq 1$ is an odd integer. Since $L = -\Delta$ is the Hodge–Laplace operator for the L^2-de Rham complex when $k = 0$, this problem can be expressed within our semilinear framework by taking $Fu = u^m$. While F is monotone (since m is odd), it does not appear to be globally Lipschitz when $m > 1$, since the inequality

$$\|Fu - Fu'\|_Y \leq C \|u - u'\|_X , \quad \forall u, u' \in X, \tag{12}$$

cannot be shown to hold for any reasonable choice of the spaces X and Y.

However, for semilinear scalar problems where both continuous and discrete maximum principles are available, it is possible to establish *a priori* L^∞ estimates on the continuous and discrete solutions. These estimates ensure that the solutions both lie in an *order interval* $[u_-, u_+] \cap \mathring{H}^1(\Omega)$ within the solution space. In other words, if u and u_h are the continuous and discrete solutions of the semilinear problem (11), then they satisfy

$$u_- \leq u, u_h \leq u_+.$$

This pointwise control makes it possible to establish (12) in this order interval, where $X = \mathring{H}^1(\Omega)$ and $Y = L^2(\Omega)$. This is precisely the Lipschitz condition that we need to apply the framework developed in this paper. In fact, even exponential-type nonlinearities can be shown to satisfy the condition (12) at the continous and discrete solutions; see, for example, [10]. For a discussion of these and related techniques for semilinear problems, see [29].

While pointwise control of the continuous solution to (11) is always available, due to the maximum principle property of the Laplacian, pointwise control of the discrete solution is in fact a much more delicate property. Typically, this requires placing restrictive angle conditions on the mesh underlying the finite element space. In two spatial dimensions, the angle conditions necessary to preserve the maximum principle property are achievable with careful mesh generation, even when local mesh refinement algorithms in are use. However, in three spatial dimensions, it is very difficult to satisfy the required angle conditions, even on quasi-uniform meshes.

Nevertheless, in the case of sub-critical and critical-type polynomial nonlinearities, it is possible to establish a *local* type of Lipschitz condition by relying only on pointwise control of the continuous solution, without requiring pointwise control of the discrete solution, and thus avoiding the need for mesh conditions altogether. For this class of nonlinearities, one can obtain the following local Lipschitz result.

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^n$ for $n \geq 2$, and assume that $\|u\|_{L^\infty(\Omega)} < \infty$. Let $F\colon \mathring{H}^1(\Omega) \to H^{-1}(\Omega)$ be a polynomial in u with measurable coefficients defined on Ω, and whose polynomial degree m satisfies $1 \leq m < \infty$ for $n = 2$ and $1 \leq m \leq \overline{m} = (n+2)/(n-2)$ for $n > 2$. Assume also that $u, u' \in \mathring{H}^1(\Omega)$, and that $\|u - u'\|_{\mathring{H}^1(\Omega)} \leq M$ for some finite constant M. Then*

$$\|Fu - Fu'\|_{H^{-1}(\Omega)} \leq C\|u - u'\|_{\mathring{H}^1(\Omega)},$$

where $C = C(\Omega, F, \|u\|_{L^\infty(\Omega)}, n, m, M)$.

Proof. See [5]. □

We note that the result in Theorem 4.5 has a slightly different form than that considered above, since $F\colon \mathring{H}^1(\Omega) \to H^{-1}(\Omega)$ rather than $\mathring{H}^1(\Omega) \to L^2(\Omega)$. In the language of Hilbert complexes, that is, the codomain is given by the dual to V^k instead of W^k. However, as remarked by Arnold, Falk, and Winther [4, p. 305], the estimates of finite element exterior calculus also apply when the data is given weakly as $f \in \left(V^k\right)^*$, equipped with the sup-norm, and the analysis does not change substantially from the $f \in W^k$ case (although the solution can no longer be interpreted as giving the Hodge decomposition of f in a strong sense). Likewise, the results presented here for the semilinear problem also extend to the case of weakly-specified data, since the tools of monotone operator theory and abstract Hammerstein equations carry over without any significant modification (other than the appearance of the sup-norm in place of the W-norm, where appropriate).

Finally, many important problems contain nonlinearities satisfying the assumptions needed to establish continuous and discrete pointwise control, either by satisfying mesh conditions or by Theorem 4.5. In particular, these examples include the Yamabe problem arising in geometric analysis, and the Hamiltonian constraint equation in general relativity. For the three-dimensional case, the leading nonlinear terms for both of these problems have the form

$$Fu = au^5 + bu,$$

where $a, b \in L^\infty(\Omega)$. Since $m = 5$ equals the critical exponent $\overline{m} = (n+2)/(n-2)$ when $n = 3$, the nonlinearity satisfies the hypotheses of Theorem 4.5. See [23] for the derivation of pointwise bounds for both problems, using maximum principles.

5. Conclusion

In this article, we have extended the abstract Hilbert complex framework of Arnold, Falk, and Winther [4], as well as our previous analysis of variational crimes from Holst and Stern [24], to a class of semilinear mixed variational problems. Our approach used an equivalent formulation of these problems as abstract Hammerstein equations, enabling us to apply the tools of nonlinear functional analysis and monotone operator theory, and to obtain well-posedness results for both continuous and discrete semilinear problems. Additional continuity assumptions on the nonlinearity yielded a stronger well-posedness result for mixed problems, as well as *a priori* error bounds for the discrete solution. Despite the addition of nonlinear terms, this result agrees with the quasi-optimal estimate of Arnold, Falk, and Winther [4] for the linear case, and similarly allows for improved estimates to be obtained under additional compactness and continuity assumptions. Likewise, in extending the variational crimes analysis in [24] to semilinear problems, we obtain convergence results agreeing with the linear case. These last results can also be used to extend the *a priori* estimates for Galerkin solutions to the Laplace–Beltrami equation on approximate 2- and 3-hypersurfaces, due to Dziuk [17] and Demlow [15], to the larger class of semilinear problems involving the Hodge Laplacian on hypersurfaces of arbitrary dimension.

At the conclusion of Holst and Stern [24], several open problems are mentioned, including the extension of the Hilbert complex framework to more general Banach complexes. While the Hilbert complex framework was again sufficient for the analysis of semilinear problems presented here, Banach spaces become necessary when dealing with more general nonlinear problems. Banach complexes appear to lack much of the crucial structure of Hilbert complexes, particularly the Hodge decomposition, whose orthogonality depends fundamentally on the presence of an inner product. However, if there is additional structure present in a Banach complex, such as a Gelfand-like triple structure (e.g., $W \subset H \subset W^*$, where H is a Hilbert complex), then it may be possible to generalize the approach taken here.

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