Overview of the Non-CMC Analysis Frameworks for the Conformal Method

Fields Institute Lecture

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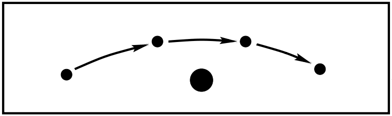
Outline (Starting Part 1)

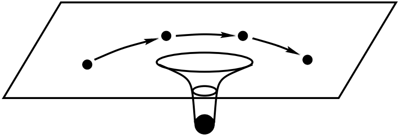
General Relativity

Einstein's general theory of relativity states that spacetime has the structure of a pseudo-Riemannian 4-manifold $\mathcal{M}$.

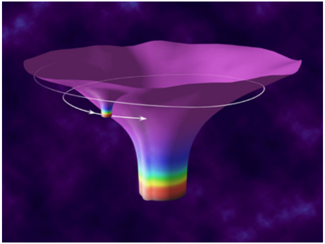
The theory predicts that accelerating masses produce gravitational waves of perturbations in the metric tensor.

Newtonian vs. General Relativistic pictures:





This space-time bending is governed by the *Einstein Equations*.
Black-Hole merger depiction (shamelessly stolen from LIGO website):



LIGO

LIGO (Laser Interferometer Gravitational-wave Observatory) is one of several recently constructed gravitational detectors.

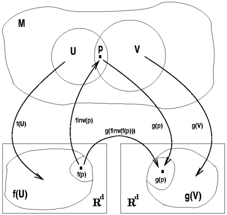
The design of LIGO is based on measuring distance changes between objects in perpendicular directions as the ripple in the metric tensor propagates through the device.

The two L-shaped LIGO observatories (in Washington and Louisiana), with legs at 1.5m meters by 4km, have phenomenal sensitivity, on the order of 10^{-15}m to 10^{-18}m.

The LIGO arms in Louisiana and Hanford, Washington:

The Einstein Equations

Riemann (curvature) tensor $R_{abc}{}^{d}$ arises as failure of commutativity of covariant differentiation:



$$\text{Flat:} \quad V^a{}_{,bc} - V^a{}_{,cb} = 0, \quad V^a{}_{,b} = \frac{\partial V^a}{\partial x^b}.$$

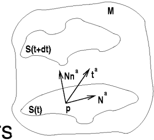
$$\text{Curved:} \; V^a{}_{;bc} - V^a{}_{;cb} = R^a{}_{dbc} V^d, \quad V^a{}_{;b} = V^a{}_{,b} + \Gamma^a_{bc} V^c,$$

where

$$R^a{}_{dbc} = \Gamma^a_{bd,c} - \Gamma^a_{cd,b} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{eb}\Gamma^e_{cd}.$$

The ten equations for the ten independent components of the symmetric spacetime metric tensor g_{ab} are the *Einstein Equations*: $G_{ab} = \kappa T_{ab}, \quad 0 \leqslant a \leqslant b \leqslant 3, \quad \kappa = 8\pi G/c^4$,

- $R_{abc}{}^{d}$; Riemann (curvature) tensor
- $R_{ab} = R_{acb}{}^{c}$, $R = R_a{}^{a}$; Ricci tensor; scalar curvature
- $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$, T_{ab}; Einstein/stress-energy tensors



Initial-value formulations well-posed (cf. Hawking & Ellis); Various formalisms yield constrained (weakly/strongly/symmetric) hyperbolic evolution systems on space-like 3-manifolds $\mathcal{S}(t)$ for a Riemannian $\hat{h}_{ab}$, possibly also extrinsic curvature $\hat{k}_{ab} \sim \frac{d}{dt}\hat{h}_{ab}$.

Einstein Constraints and Conformal Method

Twelve-component Einstein evolution system for $(\hat{h}_{ab}, \hat{k}_{ab})$ on a foliation.

Constrained by coupled eqns on spacelike $\mathcal{M} = \mathcal{M}_t$, with $\hat{\tau} = \hat{k}_{ab}\hat{h}^{ab}$,

$$ {}^3\hat{R} + \hat{\tau}^2 - \hat{k}_{ab}\hat{k}^{ab} - 2\kappa\hat{\rho} = 0, \qquad \hat{\nabla}^a\hat{\tau} - \hat{\nabla}_b\hat{k}^{ab} - \kappa\hat{j}^a = 0. $$

York conformal decomposition: split initial data into 8 freely specifiable pieces plus 4 determined via: $\hat{h}_{ab} = \phi^4 h_{ab}$, $\hat{\tau} = \hat{k}_{ab}\hat{h}^{ab} = \tau$, and

$$ \hat{k}_{ab} = \phi^{-10}[\sigma^{ab} + (\mathcal{L}w)^{ab}] + \frac{1}{4}\phi^{-4}\tau h^{ab}, \quad \hat{j}^a = \phi^{-10}j^a, \quad \hat{\rho} = \phi^{-8}\rho. $$

Produces coupled elliptic system for conformal factor ϕ and a w^a:

$$ -8\Delta\phi + R\phi + \frac{2}{3}\tau^2\phi^5 - (\sigma_{ab} + (\mathcal{L}w)_{ab})(\sigma^{ab} + (\mathcal{L}w)^{ab})\phi^{-7} - 2\kappa\rho\phi^{-3} = 0, $$

$$ -\nabla_a(\mathcal{L}w)^{ab} + \frac{2}{3}\phi^6\nabla^b\tau + \kappa j^b = 0. $$

Differential structure on $\mathcal{M}$ defined through background 3-metric h_{ab}:

$$ (\mathcal{L}w)^{ab} = \nabla^a w^b + \nabla^b w^a - \frac{2}{3}(\nabla_c w^c)h^{ab}, \qquad \nabla_b V^a = V^a_{\ ;b} = V^a_{\ ,b} + \Gamma^a_{bc}V^c, $$

$$ V^a_{\ ,b} = \frac{\partial V^a}{\partial x^b}, \qquad \Gamma^a_{bc} = \frac{1}{2}h^{ad}\left(\frac{\partial h_{db}}{\partial x^c} + \frac{\partial h_{dc}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^d}\right). \qquad (\Gamma^a_{bc} = \Gamma^a_{cb}) $$

The Conformal Method

Lichnerowicz and Choquet-Bruhat Papers: 1944 and 1958

- A. Lichnerowicz. L'integration des équations de la gravitation relativiste et le problème des n corps. *J. Math. Pures Appl.*, 23:37–63, 1944.
- Y. Choquet-Bruhat. Théorèm d'existénce en mécanique des fluides relativistes. *Bull. Soc. Math. France*, 86:155–175, 1958.

Some Key Conformal Method Papers: 1971–2014

- J. York. Gravitational degrees of freedom and the initial-value problem. *Phys. Rev. Lett.*, 26(26):1656–1658, 1971.
- J. York. Conformally invariant orthogonal decomposition of symmetric tensor on Riemannian manifolds and the initial-value problem of general relativity. *J. Math. Phys.*, 14(4):456–464, 1973.
- J. York. Conformal "thin-sandwich" data for the initial-value problem of general relativity. *Phys. Rev. Lett.*, 82:1350–1353, 1999.
- H. Pfeiffer and J. York, Jr. Extrinsic curvature and the Einstein constraints. *Phys. Rev. D*, 67:044022, 2003.
- D. Maxwell. The conformal method and the conformal thin-sandwich method are the same. arXiv:1402.5585v2, 2014.

The Conformal Method as an Elliptic System

Let $\mathcal{M}$ be a space-like Riemannian 3-manifold with (possibly empty) boundary submanifold $\partial\mathcal{M}$, split into disjoint submanifolds satisfying:

$$\partial_D\mathcal{M} \cup \partial_N\mathcal{M} = \partial\mathcal{M}, \qquad \partial_D\mathcal{M} \cap \partial_N\mathcal{M} = \emptyset. \qquad (\overline{\partial_D\mathcal{M}} \cap \overline{\partial_N\mathcal{M}} = \emptyset)$$

Metric h_{ab} associated with $\mathcal{M}$ induces boundary metric σ_{ab}, giving boundary value formulation of conformal method for ϕ and w^a:

$$\begin{aligned}
L\phi + F(\phi, w) &= 0, \ \text{ in } \mathcal{M}, && \text{(Hamiltonian)} \\
\mathbb{L}w + \mathbb{F}(\phi) &= 0, \ \text{ in } \mathcal{M}, && \text{(Momentum)} \\
(\mathcal{L}w)^{ab}\nu_b + C^a{}_b w^b &= V^a_\phi \ \text{ on } \partial_N\mathcal{M}, \quad \textit{and} && w^a = w^a_D \ \text{ on } \partial_D\mathcal{M}, \\
(\nabla^a\phi)\nu_a + k_w(\phi) &= g \ \text{ on } \partial_N\mathcal{M}, \quad \textit{and} && \phi = \phi_D \ \text{ on } \partial_D\mathcal{M},
\end{aligned}$$

where:

$$L\phi = -\Delta\phi, \quad (\mathbb{L}w)^a = -\nabla_b(\mathcal{L}w)^{ab},$$

$$F(\phi, w) = a_R\phi + a_\tau\phi^5 - a_w\phi^{-7} - a_\rho\phi^{-3}, \quad \mathbb{F}(\phi) = b^b_\tau\phi^6 + b^b_j,$$

with:

$$a_R = \tfrac{R}{8}, \ \ a_\tau = \tfrac{\tau^2}{12}, \ \ a_w = \tfrac{1}{8}[\sigma_{ab} + (\mathcal{L}w)_{ab}]^2, \ \ a_\rho = \tfrac{\kappa\rho}{4}, \ \ b^b_\tau = \tfrac{2}{3}\nabla^b\tau, \ \ b^b_j = \kappa j^b,$$

$$(\mathcal{L}w)^{ab} = \nabla^a w^b + \nabla^b w^a - \tfrac{2}{3}(\nabla_c w^c)h^{ab}, \qquad \nabla_b V^a = V^a{}_{;b} = V^a{}_{,b} + \Gamma^a_{bc}V^c,$$

$$V^a{}_{,b} = \frac{\partial V^a}{\partial x^b}, \qquad \Gamma^a_{bc} = \frac{1}{2}h^{ad}\left(\frac{\partial h_{db}}{\partial x^c} + \frac{\partial h_{dc}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^d}\right). \qquad (\Gamma^a_{bc} = \Gamma^a_{cb})$$

Well-posedness, estimates, approximation, ...

This problem has the form:

$$\text{Find } u \in \bar{u} + X \text{ such that } \langle F(u), v \rangle = 0, \ \forall v \in Y, \tag{1}$$

where X and Y are B-spaces and $F : X \to Y^*$.

Given approximation $u^0 \approx u$, Newton iteration for u has form:

(a) Find $w \in X$ such that: $\langle F'(u^k)w, v \rangle = -\langle F(u^k), v \rangle + r, \ \ \forall v \in Y$
(b) Set: $u^{k+1} = u^k + \lambda w$

One discretizes (a)-(b) at "last moment" using your favorite method.

Many questions about the constraint (and evolution) eqns remain open:

1. Is there existence, uniqueness, stability?
2. Is there multiplicity, with folds or bifurcation phenomena?
3. How *smooth* is X?
4. Can one build good approximation spaces $X_h \approx X$?
5. Performance of linear approximation for (1)?
6. Performance of nonlinear approximation for (1)?
7. Can we produce such (linear and nonlinear) approximations with optimal (linear) space and time complexity?

Outline (Starting Part 2)

The 1973–1995 CMC Results

$\nabla^b \tau = 0$: Constant Mean Curvature (CMC): $\Rightarrow$ constraints de-couple.

There were a number of CMC results generated during 1973–1995 by exploiting the fact that the constraint equations decouple.

You can solve the momentum constraint equation once and for all, and then you solve the Hamiltonian constraint once.

The research came down to understanding under what conditions the Hamiltonian constraint was solvable.

Some Key CMC Papers: 1974–1995

- N. Ó. Murchadha and J. York. Initial-value problem of general relativity I. General formulation and physical interpretation. *Phys. Rev. D*, 10(2):428–436, 1974.
- N. Ó. Murchadha and J. York. Initial-value problem of general relativity II. Stability of solution of the initial-value equations. *Phys. Rev. D*, 10(2):437–446, 1974.
- J. Isenberg. Constant mean curvature solution of the Einstein constraint equations on closed manifold. Class. Quantum Grav. **12** (1995), 2249–2274.

The 1996–2007 Near-CMC Results

$\nabla^b \tau \neq 0$: Non-CMC case: $\Rightarrow$ constraints couple.

In the Non-CMC case, the constraints couple together; through 1996 there were no results, until the Isenberg-Moncrief paper of 1996 under *near-CMC conditions* (to be explained). This led to several results.

Some of the Near-CMC Papers: 1996–2007

- J. Isenberg and V. Moncrief, *A set of nonconstant mean curvature solution of the Einstein constraint equations on closed manifolds*, Class. Quantum Grav. **13** (1996), 1819–1847.
- J. Isenberg and J. Park. Asymptotically hyperbolic non-constant mean curvature solutions of the Einstein constraint equations. *Class. Quantum Grav.*, 14:A189–A201, 1997.
- Y. Choquet-Bruhat, J. Isenberg, and J. York. Einstein constraint on asymptotically Euclidean manifolds. *Phys. Rev. D*, 61:084034, 2000.
- P. Allen, A. Clausen, and J. Isenberg. Near-constant mean curvature solutions of the Einstein constraint equations with non-negative Yamabe metrics. Available as arXiv:0710.0725 [gr-qc], 2007.

A Look at the 1996 Near-CMC Result

Fixed-point arguments involve composition $G(\phi) = T(\phi, S(\phi))$, where:

1. Given ϕ, solve MC for w: $\quad w = S(\phi)$
2. Given w, solve HC for ϕ: $\quad \phi = T(\phi, w)$

Map $S : X \to \mathcal{R}(S) \subset Y$ is MC solution map;
Map $T : X \times \mathcal{R}(S) \to X$ is some fixed-point map for HC.

Theorem: *(Isenberg-Moncrief) For case $R = -1$ on a closed manifold ($h_{ab} \in \mathcal{Y}^-$), strong smoothness assumptions, and near-CMC conditions, Isenberg-Moncrief show this is a contraction in Hölder spaces:*

$$[\phi^{(k+1)}, w^{(k+1)}] = G([\phi^{(k)}, w^{(k)}]).$$

Proof Outline: *Maximum principles, barriers, Banach algebra properties, plus contraction-mapping argument.* □

Theorem 1 (Contraction Mapping Theorem)

Let X be Banach and $U \subset X$ nonempty & closed. If $G : X \to X$ is a k-contraction on U:

$$\|G(u) - G(v)\|_X \leq k\|u - v\|_X, \quad 0 \leqslant k < 1, \quad \forall u, v \in U,$$

then there exists a (unique) fixed-point $u \in U \subset X$ satisfying $u = G(u)$.

Yamabe Classes: Closed, Smooth or Rough

Yamabe classification of smooth metrics: Let $u > 0$ solve: $-8\Delta u + Ru = R_u u^5$. Then:

$$R_u > 0 \Rightarrow h_{ab} \in \mathcal{Y}^+, \quad R_u < 0 \Rightarrow h_{ab} \in \mathcal{Y}^-, \quad R_u = 0 \Rightarrow h_{ab} \in \mathcal{Y}^0.$$

Yamabe classification of rough metrics: The Yamabe problem on closed manifolds for rough metrics is still open; however, one can still get the following result [HNT09] which is all we need here:

Theorem 2 (Yamabe Classification of Rough Metrics)

Let $(\mathcal{M}, h)$ be a smooth, closed, connected Riemannian manifold with dimension $n \geqslant 3$ and with a metric $h \in W^{s,p}$, where we assume $sp > n$ and $s \geqslant 1$. Then, the followings hold:

- $\mu_{2^\star} > 0$ *iff there is a metric in $[h]$ with continuous positive scalar curvature.*
- $\mu_{2^\star} = 0$ *iff there is a metric in $[h]$ with vanishing scalar curvature.*
- $\mu_{2^\star} < 0$ *iff there is a metric in $[h]$ with continuous negative scalar curvature.*

In particular, two conformally equivalent metrics cannot have scalar curvatures with distinct signs.

Impact of the near-CMC restriction

To establish contraction properties for coupled PDE systems gives coupling restrictions; for the constraints, the restriction that results is the near-CMC condition:

$$\|\nabla\tau\|_r < C \inf_{\mathcal{M}} |\tau|, \tag{2}$$

where particular L^r norm depends on context. Condition appears in two distinct places:

(1) Construction of the contraction G,

(2) Construction of the set U on which G is a contraction.

The near-CMC condition is basically a condition that ensures the coupling between the two equations is weak.

In [HNT08, HNT09, Max09], a non-CMC analysis framework was developed by relacing contraction argument with a Schauder argument, combined with construction of *global barriers*.

The framework in [HNT08, HNT09] required the existence of matter sources to construct sub-solutions; this was extended to vacuum (no matter sources) in [Max09], which also contains other new results.

Approach places no limit on strength of equation coupling.

The 2008 Framework: Mappings S and T

We outline the near-CMC-free fixed-point argument from [HNT09].

We first make precise the definitions of the maps S and T.

To deal with the non-trivial kernel that exists for L on closed manifolds, fix an arbitrary positive *shift* $s > 0$. Now write the constraints as

$$L_s\phi + F_s(\phi, w) = 0, \tag{3}$$

$$(\mathbb{L}w)^a + \mathbb{F}(\phi)^a = 0, \tag{4}$$

where $L_s : W^{2,p} \to L^p$ and $\mathbb{L} : W^{2,p} \to L^p$ are defined as

$$L_s\phi := [-\Delta + s]\phi, \qquad (\mathbb{L}w)^a := -\nabla_b(\mathcal{L}w)^{ab},$$

and where $F_s : [\phi_-, \phi_+] \times W^{2,p} \to L^p$ and $\mathbb{F} : [\phi_-, \phi_+] \to L^p$ are

$$F_s(\phi, w) := [a_R - s]\phi + a_\tau\phi^5 - a_w\phi^{-7} - a_\rho\phi^{-3}, \qquad \mathbb{F}(\phi)^a := b_\tau^a\phi^6 + b_j^a.$$

Introduce the operators $S : [\phi_-, \phi_+] \to W^{2,p}$ and $T : [\phi_-, \phi_+] \times W^{2,p} \to W^{2,p}$ as

$$S(\phi) := -\mathbb{L}^{-1}\mathbb{F}(\phi), \tag{5}$$

$$T(\phi, w) := -L_s^{-1}F_s(\phi, w). \tag{6}$$

Both maps are well-defined when $s > 0$ (L_s is invertible) and when there are no conformal Killing vectors ($\mathbb{L}$ is invertible).

Schauder Approach to get at Non-CMC

Alternatives to Contraction Mapping Theorem that are more topological:

Theorem 3 (Schauder Theorem)

Let X be a Banach space, and let $U \subset X$ be a non-empty, convex, closed, bounded subset. If $G : U \to U$ is a compact operator, then there exists a fixed-point $u \in U$ such that $u = G(u)$.

Here is a variation of Schauder tuned for the constraints.

Theorem 4 (Coupled Schauder Theorem)

Let X and Y be Banach spaces, and let Z be a Banach space with compact embedding $X \hookrightarrow Z$. Let $U \subset Z$ be non-empty, convex, closed, bounded, and let $S : U \to \mathcal{R}(S) \subset Y$ and $T : U \times \mathcal{R}(S) \to U \cap X$ be continuous maps. Then, there exist $w \in \mathcal{R}(S)$ and $\phi \in U \cap X$ such that

$$\phi = T(\phi, w) \quad \text{and} \quad w = S(\phi). \tag{7}$$

Proof Outline: *Show $G(\phi) = i \circ T(\phi, S(\phi)) : U \subset Z \to U \subset Z$ is compact and then use Schauder, where $i : X \to Z$ is (compact) canonical injection.* $\square$

Global barriers and *a priori* L^∞-bounds

To remove the near-CMC condition we use the following approach:

- Compactness-type fixed-point arguments (Coupled Schauder).
- Identifying the non-empty, convex, closed, bounded set U.
- Establishing properties of the constraint maps S and T.

Note: Establishing continuity of maps S and T, identifying the set U, and establishing convergence/optimality of numerical methods, will ALL depend on construction of compatible global barriers ϕ_- and ϕ_+ that are free of the near-CMC condition. (Compatibility: $0 \leqslant \phi_- \leqslant \phi_+$)

Sub- and super-solutions, or barriers to HC satisfy:

$$-\Delta\phi_- + a_R\phi_- + a_\tau\,\phi_-^5 - a_w\,\phi_-^{-7} - a_\rho\,\phi_-^{-3} \leqslant 0,$$
$$-\Delta\phi_+ + a_R\phi_+ + a_\tau\,\phi_+^5 - a_w\,\phi_+^{-7} - a_\rho\,\phi_+^{-3} \geqslant 0.$$

Barriers related to *a priori* L^∞-bounds on any solution (if one exists):

$$0 < \alpha \leqslant \phi \leqslant \beta < \infty.$$

When nonlinearity monotone decreasing, can show barriers also *a priori* L^∞-bounds. (One can establish bounds directly; see [HNT09].)

Working in ordered Banach spaces; need for non-empty order-cone interval $U = [\phi_-, \phi_+]$ leads to concept of global barriers: Barriers for HC for any a_w generated from solutions w to MC with source $\phi \in [\phi_-, \phi_+]$.

Existence/estimates for momentum constraint

Assume for the moment we have global barriers (must still find them), and they give us (must verify) a non-empty, convex, closed, bounded subset $U \subset Z$ of the Banach space Z, and that in addition, we can show (must verify) that T is invariant on U.

To use the Coupled Schauder Theorem to establish existence, it would remain to establish continuity properties of momentum and Hamiltonian constraint mappings S and T. First consider S (see [HNT09]).

Theorem 5 (MC – Existence and Estimates)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, closed, C^2, Riemannian manifold, with h_{ab} having no conformal Killing vectors, and let $b^a_\tau, b^a_j \in L^p$ with $p \geqslant 2$ and $\phi \in L^\infty$; Then, equation (4) has a unique solution $w^a \in W^{2,p}$ with

$$c\,\|w\|_{2,p} \leqslant \|\phi\|^6_\infty \,\|b_\tau\|_p + \|b_j\|_p, \tag{8}$$

where $c > 0$ is a constant.

Proof Outline: *Korn inequalities (Gårding) + Riesz-Schauder theory.* □

Generalizations appear in [HNT09], allowing rougher metric and coefficients, giving existence down to $w^a \in W^{1,p}$, with real $p \geqslant 2$.

Key inequalities for momentum constraint

Under the assumption that any $\phi \in L^\infty$ appearing as the source in the momentum constraint equation (4) satisfies for some compatible barriers $0 < \phi_- \leqslant \phi_+ < \infty$

$$\phi \in U = [\phi_-, \phi_+] \subset L^\infty,$$

then one can establish continuity of S (see [HNT09]). One can also show stronger boundedness and Lipschitz properties:

$$\|S(\phi)\|_Y \leqslant C_{SB}, \qquad \|S(\phi_1) - S(\phi_2)\|_Y \leqslant C_{SL}\|\phi_1 - \phi_2\|_Z,$$
$$Y = W^{2,p}, \qquad Z = L^\infty.$$

The inequality in equation (8) also gives for $p > 3$ the following estimate:

$$a_{\mathbf{w}} \leqslant K_1 \, \|\phi\|_\infty^{12} + K_2, \tag{9}$$

with $K_1 = (\frac{c_s c_{\mathcal{L}}}{\sqrt{2c}})^2 \|b_\tau\|_p^2$, $K_2 = \frac{1}{4}\|\sigma\|_\infty^2 + (\frac{c_s c_{\mathcal{L}}}{\sqrt{2c}})^2 \|b_j\|_p^2$, where c_s is the constant in the embedding $W^{1,p} \hookrightarrow L^\infty$, and $c_{\mathcal{L}}$ is a bound on the norm of $\mathcal{L} : W^{2,p} \to W^{1,p}$.

Inequality (9) will appear in a critical part of the analysis of the coupling between the two equations. Note that there is no smallness assumption on $\|b_\tau\|_p$, so the near-CMC condition is not required for these results.

Turn now to Hamiltonian map T. From e.g. [HNT09] we have

Theorem 6 (HC – Existence and Estimates)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, C^2, closed Riemannian manifold. Let free data τ^2, σ^2 and ρ be in L^p, with $p \geqslant 2$. Let ϕ_- and ϕ_+ be barriers to (3) for particular vector $w^a \in W^{1,2p}$. Then, there exists solution $\phi \in [\phi_-, \phi_+] \cap W^{2,p}$ of HC (3). Furthermore, if metric h_{ab} in positive Yamabe class, then ϕ is unique.

Proof Outline: *Barriers plus monotone increasing maps.* $\square$

Generalizations appear in [HNT09], allowing rougher metric and coefficients, giving existence down to $\phi \in W^{1,p}$, with real $p \geqslant 2$.

This result, together MC results above and barrier results below, give required continuity properties for map T (see [HNT09] for details). One can show stronger boundedness and Lipschitz conditions:

$$\|T(\phi, w)\|_X \leqslant C_{TB}, \qquad \|T(\phi_1, w) - T(\phi_2, w)\|_X \leqslant C_{TL}\|\phi_1 - \phi_2\|_Z,$$
$$\|T(\phi, w_1) - T(\phi, w_2)\|_X \leqslant C_{TLW}\|w_1 - w_2\|_Y,$$
$$X = W^{2,p}, Y = W^{2,p}, \qquad Z = L^\infty.$$

Construction of the nonempty closed set U

Remaining assumptions for use of the Coupled Schauder Theorem are

(A) Let $U \subset Z$ be non-empty, convex, closed, and bounded (w.r.t. vector space, topological space, normed space structure of Z).

(B) T is invariant on U.

We take $U = [\phi_-, \phi_+]_{t,q} \cap \overline{B}_R(0)$, for appropriate $t \geqslant 0$, $1 \leqslant q \leqslant \infty$, where $\overline{B}_R(0)$ is closed ball in Z of radius R about 0, and verify (A).

For brevity denote $[\phi_-, \phi_+]_q = [\phi_-, \phi_+]_{0,q}$, and $[\phi_-, \phi_+] = [\phi_-, \phi_+]_{0,\infty}$.

Lemma 7 (Order cone intervals in $W^{t,q}$)

For $t \geqslant 0$, $1 \leqslant p \leqslant \infty$, the set

$$U = [\phi_-, \phi_+]_{t,q} = \{\phi \in W^{t,q} : \phi_- \leqslant \phi \leqslant \phi_+\} \subset W^{t,q}$$

is convex with respect to the vector space structure of $W^{t,q}$ and closed in the topology of $W^{t,q}$. For $t = 0$, $1 \leqslant p \leqslant \infty$, the set U is also bounded with respect to the metric space structure of $L^q = W^{0,q}$.

Proof Outline: *Convexity straightforward; closedness follows since norm convergence in L^q, $1 \leqslant q \leqslant \infty$, implies pointwise subsequential convergence a.e., and from continuous embedding $W^{t,q} \hookrightarrow L^q$ for $t > 0$; boundedness when $t = 0$ since order cone L^q_+ is normal.* $\square$

Invariance of T on U

"Global" property of barriers ensures T invariant on $[\phi_-, \phi_+]_{\tilde{s},\tilde{p}}$. Barrier compatibility ensures interval non-empty, convex, and closed.

In smooth case can take $s = 0$, then $U = [\phi_-, \phi_+]_{0,\tilde{p}}$ bounded, since order cone structure on $L^{\tilde{p}}$ is normal.

In weak metric case $h_{ab} \in W^{s,p}$, S and T not continuous for $Z = L^\infty$, and must take $Z = W^{\tilde{s},\tilde{p}}$ to get continuity of S and T, then deal with non-normal order structure on Z. (closed intervals not bounded).

For $\tilde{s} > 0$, must then take $U = [\phi_-, \phi_+]_{\tilde{s},\tilde{p}} \cap \overline{B}_R$ to ensure U is bounded, where $\overline{B}_R$ is the closed ball in Z of radius R.

It remains then only to establish invariance of T on $\overline{B}_R$.

Lemma 8 (Invariance of T on $\overline{B}_R$.)

Assume $p \in (\frac{3}{2}, \infty)$, $s \in (\frac{3}{p}, \infty)$, that $a_{\mathbf{w}} \in W^{s-2,p}$, and that "suitable conditions" on the other data hold. Then, for any $\tilde{s} \in (\frac{3}{p}, s]$ and for some $t \in (\frac{3}{p}, \tilde{s})$ there exists a closed ball $\overline{B}_R \subset W^{\tilde{s},p}$ of radius $R = \mathcal{O}\left([1 + \|a_{\mathbf{w}}\|_{s-2,p}]^{\tilde{s}/(\tilde{s}-t)}\right)$, such that

$$\phi \in [\phi_-, \phi_+]_{\tilde{s},p} \cap \overline{B}_M \quad \Rightarrow \quad T^s(\phi, a_{\mathbf{w}}) \in \overline{B}_M.$$

Main 2008 Result: Non-CMC $W^{2,p}$ solutions

Except barrier construction (must still find them), all results in place for applying Coupled Schauder Theorem to constraints. Next (smooth) result from [HNT08]; more general result from [HNT09] after.

Theorem 9 (Non-CMC existence without near-CMC)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, smooth, closed Riemannian manifold with metric h_{ab} in positive Yamabe class with no conformal Killing vectors. Let $\tau \in W^{1,p}$, with σ^2, j^a and ρ in L^p, with $p > 3$ and small enough norms as given in Global Super-Solution Lemma so global barriers ϕ_- and ϕ_+ exist for HC (3), with $\rho \not\equiv 0$. Then, there exists $\phi \in [\phi_-, \phi_+] \cap W^{2,p}$ and $w^a \in W^{2,p}$ solving constraint equations (3)-(4).

Proof Outline: We have the operators $S : [\phi_-, \phi_+] \to W^{2,p}$ and $T : [\phi_-, \phi_+] \times W^{2,p} \to W^{2,p}$ which are again given by

$$S(\phi) := -\mathbb{L}^{-1}\mathbb{F}(\phi), \quad T(\phi, w) := -L_s^{-1} F_s(\phi, w).$$

Note the mapping S is well-defined due to absence of conformal Killing vectors, ensuring $\mathbb{L}$ is invertible. Mapping T well-defined by use of positive shift $s > 0$, ensuring L_s also invertible (see [HNT09]).

Proof outline (continued)

The constraint equations in (3)–(4) thus have precisely the form (7) for use of the Coupled Schauder Theorem.

We have the reflexive Banach spaces $X = W^{2,p}$ and $Y = W^{2,p}$, and ordered Banach space $Z = L^\infty$ with normal order cone and compact embedding $W^{2,p} \hookrightarrow L^\infty$.

With our compatible barriers forming the L^∞-interval $U = [\phi_-, \phi_+]$, we have by construction that U is non-empty as a subset of L^p, for $1 \leqslant p \leqslant \infty$. As noted earlier, the interval $[\phi_-, \phi_+] \subset L^p$ is convex with respect to the vector space structure of L^p, closed in the topology of L^p, and bounded in the norm on L^p, for $1 \leqslant p \leqslant \infty$ (see [HNT09]).

It remains to show that S and T are continuous maps from their respective domains to their respective ranges, and that T is invariant on U. These properties follow from equation (8) and from the Hamiltonian constraint theorem, with global barriers from the Global barriers theorem, using standard inequalities. The result now follows from the Coupled Schauder Theorem. $\square$

Sub-/super-solutions and *a priori* L^∞-bounds

Proofs of the results existence results were based on:

- Compactness-type fixed-point arguments (Coupled Schauder).
- Identifying a non-empty, convex, closed, bounded set U.
- Establishing continuity properties of constraint maps S and T.

Establishing continuity of maps S and T, identifying the set U, and establishing convergence/optimality of numerical methods, all depend on construction of compatible global barriers ϕ_- and ϕ_+ that are free of the near-CMC condition. (Compatibility: $0 \leqslant \phi_- \leqslant \phi_+$)

Sub- and super-solutions, or barriers to HC satisfy:

$$-\Delta\phi_- + a_R\phi_- + a_\tau\,\phi_-^5 - a_w\,\phi_-^{-7} - a_\rho\,\phi_-^{-3} \leqslant 0,$$
$$-\Delta\phi_+ + a_R\phi_+ + a_\tau\,\phi_+^5 - a_w\,\phi_+^{-7} - a_\rho\,\phi_+^{-3} \geqslant 0.$$

Barriers related to *a priori* L^∞-bounds on any solution (if one exists):

$$0 < \alpha \leqslant \phi \leqslant \beta < \infty.$$

When nonlinearity monotone decreasing, can show barriers also *a priori* L^∞-bounds. (One can establish bounds directly; see [HNT09].)

Working in ordered Banach spaces; need for non-empty order-cone interval $U = [\phi_-, \phi_+]$ leads to concept of global barriers: Barriers for HC for any a_w generated from solutions w to MC with source $\phi \in [\phi_-, \phi_+]$.

Near-CMC-free global barrier construction

Can one build Non-CMC barriers without Near-CMC conditions?

Lemma 10 (Near-CMC-Free Global Super-Solution)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, smooth, closed Riemannian manifold with metric h_{ab} in the positive Yamabe class with no conformal Killing vectors. Let u be a smooth positive solution of the Yamabe problem

$$-\Delta u + a_R u - u^5 = 0, \tag{10}$$

and define the Harnack-type constant $k = u^\wedge / u^\vee$. If the function τ is non-constant and the rescaled matter sources j^a, ρ, and traceless transverse tensor σ^{ab} are sufficiently small, then

$$\phi_+ = \epsilon u, \quad \epsilon = \left[\frac{1}{2K_1 k^{12}}\right]^{\frac{1}{4}} \tag{11}$$

is a global super-solution of the Hamiltonian constraint.

Proof Outline: Using the notation

$$E(\phi_+) = -\Delta\phi_+ + a_R\phi_+ + a_\tau\phi_+^5 - a_w\phi_+^{-7} - a_\rho\phi_+^{-3},$$

we have to show $E(\phi_+) \geqslant 0$. The definition of $\phi_+ = \epsilon u$ implies $-\Delta\phi_+ + a_R\phi_+ = \epsilon\, u^5$.

Proof outline (continued)

Using an estimate for a_w (see [HNT09]), we have then

$$E(\phi_+) \geqslant -\Delta\phi_+ + a_R\phi_+ - \frac{K_1(\phi_+^\wedge)^{12} + K_2}{\phi_+^7} - \frac{a_\rho^\wedge}{\phi_+^3}$$
$$\geqslant \epsilon\, u^5 - K_1 \left[\frac{\phi_+^\wedge}{\phi_+^\vee}\right]^{12} \phi_+^5 - \frac{K_2}{\phi_+^7} - \frac{a_\rho^\wedge}{\phi_+^3}.$$

Notice that $\phi_+^\wedge/\phi_+^\vee = u^\wedge/u^\vee = k$, therefore we have

$$E(\phi_+) \geqslant \epsilon u^5 \left[1 - K_1\, k^{12}\epsilon^4 - \frac{K_2}{\epsilon^8 u^{12}} - \frac{a_\rho^\wedge}{\epsilon^4 u^8}\right].$$

Choice of ϵ made in (11) is equivalent to condition $1/2 = 1 - K_1\, k^{12}\epsilon^4$. For this ϵ, impose on the free data σ^{ab}, ρ and j^a the condition

$$\frac{1}{2} - \frac{K_2}{\epsilon^8 (u^\vee)^{12}} - \frac{a_\rho^\wedge}{\epsilon^4 (u^\vee)^8} \geqslant 0.$$

Thus for any $K_1 > 0$, we can guarantee $E(\phi_+) \geqslant 0$. □

Remarks:
Thus global super-solutions can be built by rescaling solutions to (10).
Existence of k related to Harnack inequality for Yamabe.
Compatible global sub-solutions available so that $0 < \phi_- \leqslant \phi_+$.

Main Result 1: Non-CMC $W^{s,p}$ weak solutions

In [Max09] Maxwell extends Theorem 9 to the vacuum case.

In [HNT09] we extend Theorem 9 to rough solutions; the main results are the following three theorems.

Theorem 11 (Non-CMC $W^{s,p}$ solutions)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional closed Riemannian manifold. Let $h_{ab} \in W^{s,p}$ admit no conformal Killing field and be in $\mathcal{Y}^+(\mathcal{M})$, where $p \in (1, \infty)$ and $s \in (1 + \frac{3}{p}, \infty)$ are given. Select q and e to satisfy:

- $\frac{1}{q} \in (0,1) \cap (0, \frac{s-1}{3}) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}]$,
- $e \in (1 + \frac{3}{q}, \infty) \cap [s-1, s] \cap [\frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p}]$.

Assume that the data satisfies:

- $\tau \in W^{e-1,q}$ *if* $e \geqslant 2$, *and* $\tau \in W^{1,z}$ *otherwise, with* $z = \frac{3q}{3+\max\{0, 2-e\}q}$,
- $\sigma \in W^{e-1,q}$, *with* $\|\sigma^2\|_\infty$ *sufficiently small,*
- $\rho \in W^{s-2,p} \cap L^\infty_+ \setminus \{0\}$, *with* $\|\rho\|_\infty$ *sufficiently small,*
- $\boldsymbol{j} \in \boldsymbol{W}^{e-2,q}$, *with* $\|\boldsymbol{j}\|_{e-2,q}$ *sufficiently small.*

Then there exists $\phi \in W^{s,p}$ with $\phi > 0$ and $\boldsymbol{w} \in \boldsymbol{W}^{e,q}$ solving the constraints.

Remark: Weak metric $h_{ab} \in W^{s,p}$ requires verifying usual relationships for $W^{s,p}$ available; gives conditions on exponents s and p to ensure e.g. Laplace-Beltrami bilinear form is continuous. (Discussed at length in [HNT09, BH14].)

Main Result 2: Near-CMC $W^{s,p}$ weak solutions

Theorem 12 (Near-CMC $W^{s,p}$ solutions)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional closed Riemannian manifold. Let $h_{ab} \in W^{s,p}$ admit no conformal Killing field, where $p \in (1, \infty)$ and $s \in (1 + \frac{3}{p}, \infty)$ are given. Select q, e and z to satisfy:

- $\frac{1}{q} \in (0,1) \cap (0, \frac{s-1}{3}) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}]$,
- $e \in (1 + \frac{3}{q}, \infty) \cap [s-1, s] \cap [\frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p}]$.
- $z = \frac{3q}{3+\max\{0,2-e\}q}$.

Assume τ satisfies near-CMC condition (2) with z above, and data satisfies:

- $\tau \in W^{e-1,q}$ *if* $e > 2$, *and* $\tau \in W^{1,z}$ *if* $e \leqslant 2$,
- $\sigma \in W^{e-1,q}$,
- $\rho \in W_+^{s-2,p}$,
- $\boldsymbol{j} \in \boldsymbol{W}^{e-2,q}$.

In addition, let one of the following sets of conditions hold:

(a) h_{ab} *in* $\mathcal{Y}^-(\mathcal{M})$; h_{ab} *conformally equiv to metric w/ scalar curvature* $(-\tau^2)$;

(b) h_{ab} *in* $\mathcal{Y}^0(\mathcal{M})$ *or* $\mathcal{Y}^+(\mathcal{M})$; *either* $\rho \not\equiv 0$ *and* $\tau \not\equiv 0$ *or* $\tau \in L^\infty$ *and* $\inf_{\mathcal{M}} \sigma^2$ *suff. large.*

Then there exists $\phi \in W^{s,p}$ with $\phi > 0$ and $\boldsymbol{w} \in \boldsymbol{W}^{e,q}$ solving the constraints.

Main Result 3: CMC $W^{s,p}$ weak solutions

Theorem 13 (CMC $W^{s,p}$ solutions)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional closed Riemannian manifold. Let $h_{ab} \in W^{s,p}$ where $p \in (1,\infty)$ and $s \in (\frac{3}{p},\infty) \cap [1,\infty)$ are given. With $d := s - \frac{3}{p}$, select q and e to satisfy:

- $\frac{1}{q} \in (0,1) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}] \cap [\frac{1-d}{3}, \frac{3+sp}{6p})$,
- $e \in [1,\infty) \cap [s-1, s] \cap [\frac{3}{q} + d - 1, \frac{3}{q} + d] \cap (\frac{3}{q} + \frac{d}{2}, \infty)$.

Assume $\tau = \mathrm{const}$ (CMC) and that the data satisfies:

- $\sigma \in W^{e-1,q}$,
- $\rho \in W^{s-2,p}_{+}$,
- $\boldsymbol{j} \in \boldsymbol{W}^{e-2,q}$.

In addition, let one of the following sets of conditions hold:

(a) h_{ab} *is in* $\mathcal{Y}^{-}(\mathcal{M})$; $\tau \neq 0$;

(b) h_{ab} *is in* $\mathcal{Y}^{0}(\mathcal{M})$; $\rho \not\equiv 0$;

(c) h_{ab} *is in* $\mathcal{Y}^{+}(\mathcal{M})$; $\tau \neq 0$; $\rho \not\equiv 0$.

Then there exists $\phi \in W^{s,p}$ with $\phi > 0$ and $\boldsymbol{w} \in \boldsymbol{W}^{e,q}$ solving the Einstein constraints.

Exponent conditions for the non-CMC results

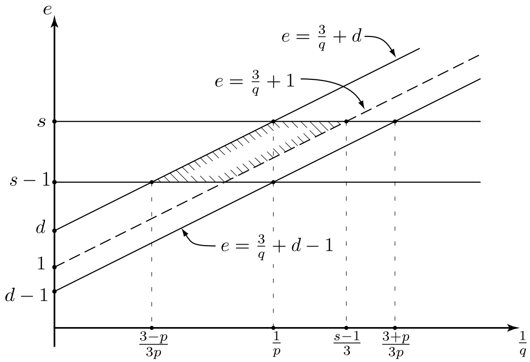


Figure: Range of e and q in Main Results 1 and 2, with $d = s - \frac{3}{p} > 1$.

Exponent conditions for the CMC results

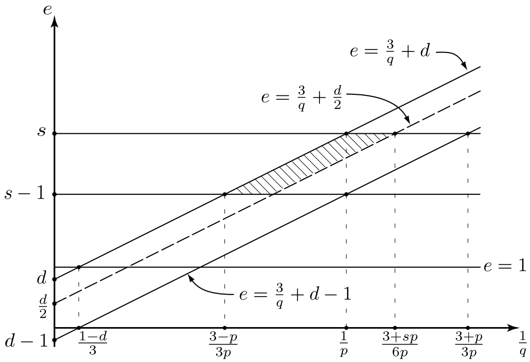


Figure: Range of e and q in Main Result 3. Recall that $d = s - \frac{3}{p} > 0$.

Prospects for other Non-CMC results

The "Schauder plus global barriers" framework in [HNT08, HNT09] has now given Non-CMC results for several other cases:

- Closed manifolds in vacuum [Max09].
- Compact manifolds with (black-hole and other) boundary [HT13, HMT14, Dilt14].
- Asymptotically Euclidean (vacuum or with black-hole inner-boundaries) [DIM14, BH14].

However, non-positive Yamabe cases present obstacle (see [HNT09]):

Lemma 14 (Near-CMC condition and $a_{\mathbf{w}}$ bounds)

Let $(\mathcal{M}, h)$ be a 3-dimensional, smooth, closed Riemannian manifold with metric $h \in W^{s,p}$ in a nonpositive Yamabe class, and let a_τ be continuous. Let $\phi_+ \in W^{s,p}$ with $\phi_+ > 0$ be a global super-solution to HC. Assume any vector field $\mathbf{w} \in \mathbf{W}^{1,2r}$ solving MC with source $\phi \leqslant \phi_+$ satisfies

$$a_{\mathbf{w}} \leqslant \theta K_1 \|\phi_+\|_\infty^{12} + \theta K_2,$$

with some positive constants θK_1 and θK_2. Moreover, assume this estimate is sharp in that for any $x \in \mathcal{M}$ there exist an open neighborhood $U \ni x$ and vector field $\mathbf{w} \in \mathbf{W}^{1,2r}$ solving MC with source $\phi \leqslant \phi_+$, such that

$$a_{\mathbf{w}} = \theta K_1 \|\phi_+\|_\infty^{12} + \theta K_2 \qquad \text{in } U. \tag{12}$$

Then, we have $\theta K_1 \leqslant \sup_{\mathcal{M}} a_\tau$.

The 2010 Limit Equation Technique

Dahl, Gicquaud, and Humbert [DGH14] study the vacuum case:

$$-2\kappa q \Delta_g \phi + R_g \phi = -\kappa \tau^2 \phi^{q-1} + |\sigma + \mathcal{L}w|^2 \phi^{-q-1}, \tag{13}$$

$$\Delta_{\mathcal{L}} w = \kappa \phi^q \, d\tau. \tag{14}$$

The notation here is:

$$(\Delta_{\mathcal{L}} w)^a = -\nabla_b (\mathcal{L}w)^{ab}, \quad q = \frac{2n}{n-2}, \quad \kappa = \frac{n-1}{n}. \tag{15}$$

They prove the following result:

Theorem 15 (Limit Equation)

Assume τ does not vanish, (M, g) have no conformal Killing vectors, and $\sigma \neq 0$ if $\mathcal{Y}(g) \geq 0$. Then at least one of the following is true:

1. *The system (13)–(14) admits a solution (ϕ, W) with $\phi > 0$.*
2. *For some $\alpha_0 \in (0, 1]$ there exists a non-trivial solution W of:*

$$\Delta_{\mathcal{L}} w = \alpha_0 \kappa |\mathcal{L}w| \, \frac{d\tau}{\tau} \qquad \textit{(limit equation)} \tag{16}$$

Important (surprising) implication: If (16) has no solution for any $\alpha_0 \in (0, 1]$, then *there must be a solution* (ϕ, W) to (13)–(14) with $\phi > 0$.

Limit Equation Insights and Limitations

It is interesting that the proof of Theorem 15 relies on use of the non-CMC analysis framework and theorems from [HNT09, Max09], but *for sub-critical exponent* to avoid need for global barriers.

Limit equation seems to offer new way to find non-CMC solutions, and has led to new insight into conformal method for non-CMC situations.

But, approach has some limitations for finding non-CMC solutions:

- It appears difficult to apply the technique outside compact case.
- The only known applications to date are near-CMC examples.

Some of the key Limit Equation papers are:

- M. Dahl, R. Gicquaud, and E. Humbert. A limit equation associated to the solvability of the vacuum Einstein constraint equations using the conformal method. arXiv:1012.2188, 2014.
- M. Dahl, R. Gicquaud, and E. Humbert. A non-existence result for a generalization of the equations of the conformal method in general relativity. arXiv:1207.5131, 2014.

2013 Implicit Function Theorem Technique

Gicquaud and Ngo [GiNg14] study again the vacuum case:

$$-2\kappa q \Delta_g \phi + R_g \phi = -\kappa \tau^2 \phi^{q-1} + |\sigma + \mathcal{L}w|^2 \phi^{-q-1}, \tag{17}$$

$$\Delta_{\mathcal{L}} w = \kappa \phi^q \, d\tau. \tag{18}$$

They prove the following non-CMC result:

Theorem 16 (Non-CMC via IFT)

Assume (M, g) have no conformal Killing vectors, $\tilde{\sigma} \neq 0$, and $\mathcal{Y}(g) > 0$. Then there exists $\eta_0 > 0$ such that for all $\eta \in (0, \eta_0)$, there exists (ϕ, W) solving (17)–(18) for $\sigma = \eta\tilde{\sigma}$.

This result appears to be the same type of general non-CMC result as those contained in the 2008 and 2009 papers [HNT08, HNT09, Max09]. The conditions are basically the same:

1. Arbitrarily prescribed mean extrinsic curvature τ.
2. No conformal Killing fields.
3. Positive Yamabe class: $\mathcal{Y}(g) > 0$.
4. Data σ "sufficiently small".

Implicit Function Theorem Technique

What is *remarkable* about the Gicquaud-Ngo Theorem 16 from [GiNg14] is that their proof goes through the *Implicit Function Theorem*, about $\tau = 0$.

In particular, they first show: There exists $\epsilon > 0$ such that the following *μ-deformed system* admits a solution $(\tilde{\phi}, \tilde{w})$ for any $\mu \in [0, \epsilon)$:

$$-2\kappa q \Delta_g \tilde{\phi} + R_g \tilde{\phi} = -\kappa \tau^2 \mu^2 \tilde{\phi}^{q-1} + |\sigma + \mathcal{L}\tilde{w}|^2 \tilde{\phi}^{-q-1}, \tag{19}$$

$$\Delta_{\mathcal{L}} \tilde{w} = \kappa \tilde{\phi}^q \mu \, d\tau. \tag{20}$$

The proof of this fact is through the Implicit Function Theorem.

They then show that $(\tilde{\phi}, \tilde{w})$ solving (19)–(20) gives a solution to (17)–(18) via the transformation:

$$\phi_\mu = \mu^{\frac{2}{q-2}} \tilde{\phi}_\mu,$$

$$w_\mu = \mu^{\frac{q+2}{q-2}} \tilde{w}_\mu,$$

$$\sigma_\mu = \mu^{\frac{q+2}{q-2}} \tilde{\sigma}_\mu,$$

$$\eta_0 = \epsilon^{\frac{q+2}{q-2}},$$

with η_0 playing its role in Theorem 16.

Conformal Method and Non-CMC: Bad News?

Does Gicquaud-Ngo Theorem 16 from [GiNg14], based on using Implicit Function Theorem about $\tau = 0$, mean that all "Far"-from-CMC results requiring small σ are effectively near-CMC after all?

Effectively yes, BUT they are not EXACTLY the same results.

The smallness conditions on σ in the 2008–2009 papers [HNT08, HNT09, Max09] are based on building global supersolutions from scaled solutions to Yamabe-type problems.

The Harnack constant for these scaled solutions, together with other constants, give specific size limits on an L^r norm of σ. I.e., σ must be "small enough", but not infinitesimally small as in the IFT arguments.

However, the distinction between these two types of "small σ" results is probably not important.

What is clear, is that the conformal method seems to have several serious problems for Non-CMC:

- Non-uniqueness for non-CMC as you move away from near-CMC.
- No arbitrary τ existence results for anything but $\mathcal{Y}(g) > 0$.
- Small σ is (effectively) near-CMC after all.

Conformal Method and Non-CMC: Bad News?

Conformal method has several serious problems for Non-CMC:

- Non-uniqueness for non-CMC as you move away from near-CMC.
- No arbitrary τ existence results for anything but $\mathcal{Y}(g) > 0$.
- Small σ is (effectively) near-CMC after all.

An alternative to the conformal method has been developed over the last several years in a sequence of papers:

- D. Maxwell. The conformal method and the conformal thin-sandwich method are the same. arXiv:1402.5585v2, 2014.
- D. Maxwell. Conformal Parameterizations of Slices of Flat Kasner Spacetimes arXiv:1404.7242v1, 2014.
- D. Maxwell. Initial data in general relativity described by expansion, conformal deformation and drift. arXiv:1407.1467, 2014.

These were based on insight gained from the multipicity result in:

- D. Maxwell. A model problem for conformal parameterizations of the Einstein constraint equations. arXiv:0909.5674, 2009.

David will tell us about some of these ideas later this week.

Outline (Starting Part 3)

Rough Metrics

[HNT09] MH, G. Nagy, and G. Tsogtgerel, *Rough solutions of the Einstein constraints on closed manifolds without near-CMC conditions*, Comm. Math. Phys. **288** (2009), no. 2, 547–613, Available as arXiv:0712.0798 [gr-qc].

[HT13] MH and G. Tsogtgerel, *The Lichnerowicz equation on compact manifolds with boundary*, Class. Quantum Grav., 30 (2013), pp. 1–31. Available as arXiv:1306.1801 [gr-qc].

[HMT14] MH, C. Meier, and G. Tsogtgerel, *Non-CMC solutions of the Einstein constraint equations on compact manifolds with apparent horizon boundaries*, Accepted for publication. Available as arXiv:1310.2302 [gr-qc].

[BH14] A. Behezadan and MH, *Rough solutions of the Einstein constraint equations on asymptotically flat manifolds without near-CMC conditions*, Preprint. Available as arXiv:1504.04661 [gr-qc].

Relevant to the study of the Einstein evolution equations is the existence of solutions to the constraint equations for weak or *rough* background metrics h_{ab}. Initial results were developed for the CMC case in [yCB04, Max05a, Max06].

Requires careful examination of multiplication properties of the spaces.

We developed Non-CMC rough solution results for closed manifolds in [HNT09], for compact manifolds with boundary in [HT13, HMT14], and for AE manifolds in [BH14].

The main results for rough non-CMC solutions on compact manifolds in [HNT09] are contained in the following three theorems.

Theorem 17 (Non-CMC $W^{s,p}$ solutions)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional closed Riemannian manifold. Let $h_{ab} \in W^{s,p}$ admit no conformal Killing field and be in $\mathcal{Y}^+(\mathcal{M})$, where $p \in (1, \infty)$ and $s \in (1 + \frac{3}{p}, \infty)$ are given. Select q and e to satisfy:

- $\frac{1}{q} \in (0,1) \cap (0, \frac{s-1}{3}) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}]$,
- $e \in (1 + \frac{3}{q}, \infty) \cap [s-1, s] \cap [\frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p}]$.

Assume that the data satisfies:

- $\tau \in W^{e-1,q}$ *if* $e \geqslant 2$, *and* $\tau \in W^{1,z}$ *otherwise, with* $z = \frac{3q}{3+\max\{0,2-e\}q}$,
- $\sigma \in W^{e-1,q}$, *with* $\|\sigma^2\|_\infty$ *sufficiently small,*
- $\rho \in W^{s-2,p} \cap L^\infty_+ \setminus \{0\}$, *with* $\|\rho\|_\infty$ *sufficiently small,*
- $\boldsymbol{j} \in \boldsymbol{W}^{e-2,q}$, *with* $\|\boldsymbol{j}\|_{e-2,q}$ *sufficiently small.*

Then there exists $\phi \in W^{s,p}$ *with* $\phi > 0$ *and* $\boldsymbol{w} \in \boldsymbol{W}^{e,q}$ *solving the constraints.*

Main Result 2: Near-CMC $W^{s,p}$ weak solutions

Theorem 18 (Near-CMC $W^{s,p}$ solutions)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional closed Riemannian manifold. Let $h_{ab} \in W^{s,p}$ admit no conformal Killing field, where $p \in (1, \infty)$ and $s \in (1 + \frac{3}{p}, \infty)$ are given. Select q, e and z to satisfy:

- $\frac{1}{q} \in (0,1) \cap (0, \frac{s-1}{3}) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}]$,
- $e \in (1 + \frac{3}{q}, \infty) \cap [s-1, s] \cap [\frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p}]$.
- $z = \frac{3q}{3+\max\{0,2-e\}q}$.

Assume τ satisfies near-CMC condition (2) with z above, and data satisfies:

- $\tau \in W^{e-1,q}$ *if* $e > 2$, *and* $\tau \in W^{1,z}$ *if* $e \leqslant 2$,
- $\sigma \in W^{e-1,q}$,
- $\rho \in W^{s-2,p}_{+}$,
- $\boldsymbol{j} \in \boldsymbol{W}^{e-2,q}$.

In addition, let one of the following sets of conditions hold:

(a) h_{ab} *in* $\mathcal{Y}^{-}(\mathcal{M})$; h_{ab} *conformally equiv to metric w/ scalar curvature* $(-\tau^2)$;

(b) h_{ab} *in* $\mathcal{Y}^{0}(\mathcal{M})$ *or* $\mathcal{Y}^{+}(\mathcal{M})$; *either* $\rho \not\equiv 0$ *and* $\tau \not\equiv 0$ *or* $\tau \in L^{\infty}$ *and* $\inf_{\mathcal{M}} \sigma^2$ *suff. large.*

Then there exists $\phi \in W^{s,p}$ with $\phi > 0$ and $\boldsymbol{w} \in \boldsymbol{W}^{e,q}$ solving the constraints.

Main Result 3: CMC $W^{s,p}$ weak solutions

Theorem 19 (CMC $W^{s,p}$ solutions)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional closed Riemannian manifold. Let $h_{ab} \in W^{s,p}$ where $p \in (1,\infty)$ and $s \in (\frac{3}{p},\infty) \cap [1,\infty)$ are given. With $d := s - \frac{3}{p}$, select q and e to satisfy:

- $\frac{1}{q} \in (0,1) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}] \cap [\frac{1-d}{3}, \frac{3+sp}{6p})$,
- $e \in [1,\infty) \cap [s-1, s] \cap [\frac{3}{q} + d - 1, \frac{3}{q} + d] \cap (\frac{3}{q} + \frac{d}{2}, \infty)$.

Assume $\tau = \mathrm{const}$ (CMC) and that the data satisfies:

- $\sigma \in W^{e-1,q}$,
- $\rho \in W^{s-2,p}_{+}$,
- $\boldsymbol{j} \in \boldsymbol{W}^{e-2,q}$.

In addition, let one of the following sets of conditions hold:

(a) h_{ab} *is in* $\mathcal{Y}^{-}(\mathcal{M})$; $\tau \neq 0$;
(b) h_{ab} *is in* $\mathcal{Y}^{0}(\mathcal{M})$; $\rho \not\equiv 0$;
(c) h_{ab} *is in* $\mathcal{Y}^{+}(\mathcal{M})$; $\tau \neq 0$; $\rho \not\equiv 0$.

Then there exists $\phi \in W^{s,p}$ with $\phi > 0$ and $\boldsymbol{w} \in \boldsymbol{W}^{e,q}$ solving the Einstein constraints.

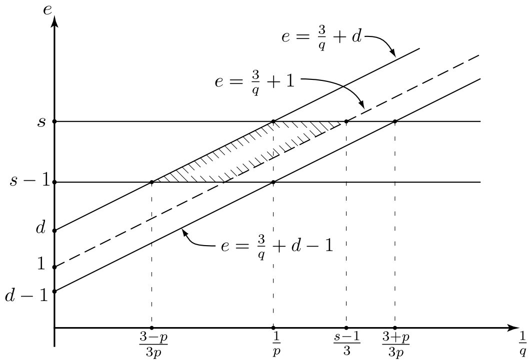


Figure: Range of e and q in Main Results 1 and 2, with $d = s - \frac{3}{p} > 1$.

Exponent conditions for the CMC results

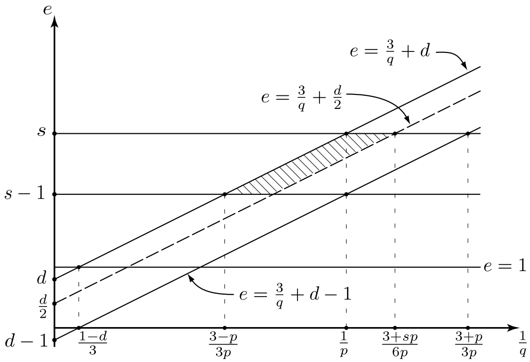


Figure: Range of e and q in Main Result 3. Recall that $d = s - \frac{3}{p} > 0$.

Really Rough Metrics

[HM13] MH and C. Meier, *Generalized solutions to semilinear elliptic PDE with applications to the Lichnerowicz equation*, Acta Appicandae Mathematicae, 130 (2014), pp. 163–203. Available as arXiv:1112.0351 [math.NA].

One of the difficulties associated with obtaining rough solutions to the conformal formulation is that the spaces $W^{s,p}(\mathcal{M})$ are not closed under multiplication unless $s > d/p$ (where d is the spatial dimension).

This restriction is a by-product of a more general problem, which is that there is no well-behaved definition of distributional multiplication that allows for the multiplication of arbitrary distributions.

Limits spaces one considers when developing weak formulation of a given elliptic partial differential equation, and places a restriction on regularity of the specified data $(g_{ab}, \tau, \sigma, \rho, j)$ of the constraint equations.

In [HM13], we extend the work of Mitrovic-Pilipovic (2006) and Pilipovic-Scarpalezos (2006) to solve problems similar to Hamiltonian constraint with distributional coefficients in *Colombeau algebras*.

These generalized spaces allows one to circumvent the restrictions associated with Sobolev coefficients and data, and thereby consider problems with coefficients and data of much lower regularity.

Compact Manifolds with Boundary

[HT13] MH and G. Tsogtgerel, *The Lichnerowicz equation on compact manifolds with boundary*, Class. Quantum Grav., 30 (2013), pp. 1–31. Available as arXiv:1306.1801 [gr-qc].

[HMT14] MH, C. Meier, and G. Tsogtgerel, *Non-CMC solutions of the Einstein constraint equations on compact manifolds with apparent horizon boundaries*, Accepted for publication. Available as arXiv:1310.2302 [gr-qc].

Compact manifolds with boundary emerge when one eliminates asymptotic ends or singularities from the manifold.

To follow closely [HT13], we change notation slightly and refer to spatial metric as g and $\hat{g}$ rather than h and $\hat{h}$.

To allow for a general discussion, assume the spatial dimension is $n \geqslant 3$; later we restrict to $n = 3$.

Let M be a compact manifold with boundary.
Let ϕ be a positive scalar field on M.
Decompose extrinsic curvature as $\hat{K} = \hat{S} + \tau\hat{g}$.
Here $\tau = \frac{1}{n}\mathrm{tr}_{\hat{g}}\hat{K}$ is (averaged) trace, so $\hat{S}$ is the traceless part.
With $\bar{q} = \frac{n}{n-2}$, conformal metric g and symmetric traceless S come via

$$\hat{g} = \phi^{2\bar{q}-2}g, \qquad \hat{S} = \phi^{-2}S. \tag{21}$$

Lichnerowicz, Compact with Boundary

Chosen powers give Lichnerowicz equation and momentum constraint:

$$-\frac{4(n-1)}{n-2}\Delta\phi + R\phi + n(n-1)\tau^2\phi^{2\bar{q}-1} - |S|_g^2\phi^{-2\bar{q}-1} = 0, \tag{22}$$

$$\mathrm{div}_g S - (n-1)\phi^{2\bar{q}}\mathrm{d}\tau = 0, \tag{23}$$

where $\Delta \equiv \Delta_g$ is the Laplace-Beltrami operator with respect to the metric g, and $R \equiv \mathrm{scal}_g$ is the scalar curvature of g.

Interpret (22)–(23) as PDE for ϕ and (part of) traceless symmetric S.

Metric g is considered as given.

To rephrase, given ϕ and S fulfilling (22)–(23), $\hat{g}$ and $\hat{K}$ given by

$$\hat{g} = \phi^{2\bar{q}-2}g, \qquad \hat{K} = \phi^{-2}S + \phi^{2\bar{q}-2}\tau g,$$

satisfy the Einstein constraint system.

$\hat{g}$ = *physical metric*
g = *conformal metric* (only specifies conformal class of $\hat{g}$, other info lost)

Assume now that traceless symmetric bilinear form S given.

Consider Lichnerowicz (22) on a compact manifold with boundary.

Boundaries emerge when one eliminates asymptotic ends or singularities from the manifold.

Need to impose appropriate boundary conditions for ϕ.

Approximating Asymptotically Flat Manifolds

On asymptotically flat manifolds, one has [YP82]

$$\phi = 1 + Ar^{2-n} + \varepsilon, \qquad \text{with} \quad \varepsilon = O(r^{1-n}), \quad \text{and} \quad \partial_r \varepsilon = O(r^{-n}), \quad (24)$$

where A is multiple total energy, r is the flat-space radial coordinate.

Idea is: cut out asymptotically Euclidean end along the sphere with large radius r and impose Dirichlet condition $\phi \equiv 1$ at boundary.

Improvement via differentiating (24) with respect to r and eliminating A:

$$\partial_r \phi + \frac{n-2}{r}(\phi - 1) = O(r^{-n}). \quad (25)$$

Equating right hand side to zero gives inhomogeneous Robin condition known to give accurate values for total energy.

Approximating Black Hole Data

Main approach: excise region around singularities and solve in exterior.

Such are "inner"-boundaries; again need boundary conditions.

In [YP82] they introduce

$$\partial_r \phi + \frac{n-2}{2a}\phi = 0, \qquad \text{for } r = a. \tag{26}$$

Means $r = a$ is a minimal surface; under appropriate data conditions minimal surface is a *trapped surface*.

Trapped surface important since implies existence of event horizon outside surface.

Various trapped surface conditions more general than minimal surface. in literature.

Make clear what we mean by a trapped surface.

Suppose all necessary regions (singularities, asymptotic ends) excised from initial slice,

Assume boundary $\Sigma := \partial M$ has finitely many components $\Sigma_1, \Sigma_2, \ldots$.

Trapped Surfaces

Let $\hat{\nu} \in \Gamma(T\Sigma^{\perp})$ be outward pointing unit normal (wrt $\hat{g}$).

Expansion scalars corresponding to outgoing and ingoing future directed null geodesics orthogonal to Σ are given by

$$\hat{\theta}_{\pm} = \mp(n-1)\hat{H} + \mathrm{tr}_{\hat{g}}\hat{K} - \hat{K}(\hat{\nu},\hat{\nu}), \qquad (27)$$

where $(n-1)\hat{H} = \mathrm{div}_{\hat{g}}\hat{\nu}$ is the mean extrinsic curvature of Σ.

Surface Σ_i is called *trapped surface* if $\hat{\theta}_{\pm} < 0$ on Σ_i.
Called *marginally trapped surface* if $\hat{\theta}_{\pm} \leqslant 0$ on Σ_i.

In terms of the conformal quantities:

$$\hat{\theta}_{\pm} = \mp(n-1)\phi^{-\bar{q}}\left(\tfrac{2}{n-2}\partial_\nu\phi + H\phi\right) + (n-1)\tau - \phi^{-2\bar{q}}S(\nu,\nu), \qquad (28)$$

where $\nu = \phi^{\bar{q}-1}\hat{\nu}$ is the unit normal with respect to g, and $\partial_\nu\phi$ is the derivative of ϕ along ν.

The mean curvature H with respect to g is related to $\hat{H}$ by

$$\hat{H} = \phi^{-\bar{q}}\left(\tfrac{2}{n-2}\partial_\nu\phi + H\phi\right). \qquad (29)$$

Trapped Surfaces: Maxwell Approach

In [Max05b, Dai04], authors studied boundary conditions leading to trapped surfaces in the asymptotically flat and constant mean curvature ($\tau = \mathrm{const}$) setting.

Decay condition on $\hat{K}$ gives automatically $\tau \equiv 0$.

In [Max05b], boundary conditions obtained via setting $\hat{\theta}_+ \equiv 0$.

More generally, if one specifies *scaled expansion scalar* $\theta_+ := \phi^{\bar{q}-e}\hat{\theta}_+$ for some $e \in \mathbb{R}$, and poses no restriction on τ, then the (inner) boundary condition for the Lichnerowicz equation (22) can be given by

$$\frac{2(n-1)}{n-2}\partial_\nu \phi + (n-1)H\phi - (n-1)\tau\phi^{\bar{q}} + S(\nu,\nu)\phi^{-\bar{q}} + \theta_+\phi^e = 0. \quad (30)$$

Trapped Surfaces: Dain Approach

In [Dai04], boundary conditions obtained via specifying $\hat{\theta}_-$.

Similarly to Maxwell case, if generalize approach so that $\theta_- := \phi^{\bar{q}-e}\hat{\theta}_-$ is specified, then we get the (inner) boundary condition

$$\frac{2(n-1)}{n-2}\partial_\nu\phi + (n-1)H\phi + (n-1)\tau\phi^{\bar{q}} - S(\nu,\nu)\phi^{-\bar{q}} - \theta_-\phi^e = 0. \quad (31)$$

Note that in above , one of $\theta_\pm$ remains unspecified, so in order to guarantee that both $\theta_\pm \leqslant 0$, one has to impose some conditions on the data, e.g., on τ or on S.

Another option: rigidly specify both $\theta_\pm$; can eliminate S from (28) and get boundary condition

$$\frac{4(n-1)}{n-2}\partial_\nu\phi + 2(n-1)H\phi + (\theta_+ - \theta_-)\phi^e = 0. \quad (32)$$

At the same time, eliminating the term involving $\partial_\nu\phi$ from (28) we get a boundary condition on S that reads as

$$2S(\nu,\nu) = 2(n-1)\tau\phi^{2\bar{q}} - (\theta_+ + \theta_-)\phi^{e+\bar{q}}. \quad (33)$$

Trapped Surfaces: A General Approach

We see something interesting: the Lichnerowicz equation couples to the momentum constraint (23) through the boundary conditions.

Even in constant mean curvature setting (where $\tau \equiv \mathrm{const}$), constraint equations (22)–(23) generally do *not* decouple.

The only reasonable way to decouple the constraints is to consider $\tau \equiv 0$ and $e = -\bar{q}$.
Note that all boundary conditions considered above (except Dirichlet) are of form:

$$\partial_\nu \phi + b_H \phi + b_\theta \phi^e + b_\tau \phi^{\bar{q}} + b_w \phi^{-\bar{q}} = 0. \tag{34}$$

Eg., in (30) and (31), one has $b_H = \frac{n-2}{2} H$, $b_\theta = \pm \frac{n-2}{2(n-1)} \theta_\pm$, $b_\tau = \mp \frac{n-2}{2} \tau$, and $b_w = \pm \frac{n-2}{2(n-1)} S(\nu, \nu)$.

Minimal surface condition (26) corresponds to the choice $b_\theta = b_\tau = b_w = 0$, and $b_H = \frac{n-2}{2} H$.

The outer Robin condition (25) is $b_H = (n-2)H$, $b_\theta = -(n-2)H$ with $e = 0$, and $b_\tau = b_w = 0$.

The Setup: General BVP for Lichnerowicz

Here we suppose each boundary component Σ_i has either Dirichlet condition $\phi \equiv 1$ or the Robin condition (34) enforced.

In particular, we allow the situation where no Dirichlet condition is imposed anywhere.

Also, to allow linear Robin condition (25) and a nonlinear condition like (30) at same time, must allow exponent e in (34) to be only locally constant.

Main tools used in paper are order-preserving maps iteration together with maximum principles and some results from conformal geometry.

These techniques sensitive to signs of coefficients in (34).

Defocusing case (preferred signs): $(e-1)b_\theta \geqslant 0$, $b_\tau \geqslant 0$, and $b_w \leqslant 0$.

Non-Defocusing case: Otherwise.

Results for defocusing case (terminology motivated by dispersive equations) more or less complete (see below).

Summary of Main Results in [HT13]

The main results and supporting tools appearing in [HT13] are:

- Justification of Yamabe classification of rough metrics on compact manifolds with boundary.
- Basic result on conformal invariance of Lichnerowicz equation.
- A uniqueness result for the Lichnerowicz equation.
- An order-preserving maps theorem for manifolds with boundary.
- Construction of upper and lower barriers that respect the trapped surface conditions.
- Combination of the results above to produce a fairly complete existence and uniqueness theory for the defocusing case.
- Combination of the results above to produce some partial results for the non-defocusing case.
- Some perturbation results (looking ahead to the asymptotically Euclidean case).
- All of the results are developed for rough (and smooth) metrics.

Yamabe Classes: Rough/Compact/Boundary

Yamabe classification of rough metrics: The Yamabe problem for rough metrics on compact manifolds with boundary is again still open; the work [Esc92, Esc96] was for smooth metrics. However, as in the closed case, one can still get the following result [HT13] which is all we need:

Theorem 20 (Yamabe Classification of Rough Metrics)

Let (M, g) be a smooth, compact, connected Riemannian manifold with boundary, where we assume that the components of the metric g are (locally) in $W^{s,p}$, with $sp > n$ and $s \geqslant 1$. Let the dimension of M be $n \geqslant 3$. Then, the following are equivalent:

a) $\mathcal{Y}_g > 0$ *(*$\mathcal{Y}_g = 0$ *or* $\mathcal{Y}_g < 0$*).*

b) $\mathcal{Y}_g(q, r, b) > 0$ *(resp.* $\mathcal{Y}_g(q, r, b) = 0$ *or* $\mathcal{Y}_g(q, r, b) < 0$*) for any* $q \in [2, 2\bar{q})$, $r \in [2, \bar{q} + 1)$ *with* $q > r$, *and any* $b \in \mathbb{R}$.

c) *There is a metric in* $[g]$ *whose scalar curvature is continuous and positive (resp. zero or negative), and boundary mean curvature is continuous and has any given sign (resp. is identically zero, has any given sign).*

In particular, two conformally equivalent metrics cannot have scalar curvatures with distinct signs.

Conformal Invariance

Let M be smooth, compact, connected n-dimensional manifold with boundary, equipped with a Riemannian metric $g \in W^{s,p}$, $n \geqslant 3$, $p \in (1, \infty)$, and that $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$.

We consider following model for Lichnerowicz problem

$$F(\phi) := \begin{pmatrix} -\Delta\phi + \frac{n-2}{4(n-1)} R\phi + a\phi^t \\ \gamma_N \partial_\nu \phi + \frac{n-2}{2} H \gamma_N \phi + b(\gamma_N \phi)^e \\ \gamma_D \phi - c \end{pmatrix} = 0,$$

where $t, e \in \mathbb{R}$ constants, $R \in W^{s-2,p}(M)$ and $H \in W^{s-1-\frac{1}{p},p}(\Sigma)$ are scalar and mean curvatures of metric g, and other coefficients satisfy $a \in W^{s-2,p}(M)$, $b \in W^{s-1-\frac{1}{p},p}(\Sigma_N)$, and $c \in W^{s-\frac{1}{p},p}(\Sigma_D)$.

Setting $\bar{q} = \frac{n}{n-2}$, interested in transformation properties of F under conformal change $\tilde{g} = \theta^{2\bar{q}-2} g$ with factor $\theta \in W^{s,p}(M)$ satisfying $\theta > 0$.

Conformal Invariance

To this end, we consider

$$\tilde{F}(\psi) := \begin{pmatrix} -\tilde{\Delta}\psi + \frac{n-2}{4(n-1)}\tilde{R}\psi + \tilde{a}\psi^t \\ \gamma_N \partial_{\tilde{\nu}}\psi + \frac{n-2}{2}\tilde{H}\gamma_N\psi + \tilde{b}(\gamma_N\psi)^e \\ \gamma_D\psi - \tilde{c} \end{pmatrix} = 0,$$

where $\tilde{\Delta}$ is Laplace-Beltrami operator associated to metric $\tilde{g}$, $\tilde{\nu}$ is the outer normal to Σ with respect to $\tilde{g}$, $\tilde{R} \in W^{s-2,p}(M)$ and $\tilde{H} \in W^{s-1-\frac{1}{p},p}(\Sigma)$ are respectively the scalar and mean curvatures of $\tilde{g}$, and $\tilde{a} \in W^{s-2,p}(M)$, $\tilde{b} \in W^{s-1-\frac{1}{p},p}(\Sigma_N)$, and $\tilde{c} \in W^{s-\frac{1}{p},p}(\Sigma_D)$.

The result we need in this direction is the following [HT13].

Lemma 21 (Conformal Invariance)

Let $\tilde{a} = \theta^{t+1-2\bar{q}}a$, $\tilde{b} = \theta^{e-\bar{q}}b$, and $\tilde{c} = \theta^{-1}c$. Then we have

$$\begin{aligned} \tilde{F}(\psi) = 0 \quad &\Leftrightarrow \quad F(\theta\psi) = 0, \\ \tilde{F}(\psi) \geqslant 0 \quad &\Leftrightarrow \quad F(\theta\psi) \geqslant 0, \\ \tilde{F}(\psi) \leqslant 0 \quad &\Leftrightarrow \quad F(\theta\psi) \leqslant 0. \end{aligned}$$

Uniqueness Results

The conformal invariance result implies the following uniqueness result for the model Lichnerowicz problem [HT13].

Lemma 22 (Uniqueness 1)

Let the coefficients of the model Lichnerowicz problem satisfy $(t-1)a \geqslant 0$, $(e-1)b \geqslant 0$, and $c > 0$. If the positive functions $\theta, \phi \in W^{s,p}(M)$ are distinct solutions of the constraint, i.e., $F(\theta) = F(\phi) = 0$, and $\theta \neq \phi$, then $(t-1)a = 0$, $(e-1)b = 0$, $\Sigma_D = \varnothing$, and the ratio θ/ϕ is constant. If in addition, $t \neq 1$, then $\mathcal{Y}_g = 0$.

The following theorem essentially says that in order to have multiple positive solutions the Lichnerowicz problem must be a linear pure Robin boundary value problem on a conformally flat manifold [HT13].

Theorem 23 (Uniqueness 2)

Let the coefficients of the Lichnerowicz problem satisfy $a_\tau \geqslant 0$, $a_w \geqslant 0$, $(e-1)b_\theta \geqslant 0$, $b_\tau \geqslant 0$, $b_w \leqslant 0$, and $\phi_D > 0$. Let the positive functions $\theta, \phi \in W^{s,p}(M)$ be solutions of the Lichnerowicz problem, with $\theta \neq \phi$. Then $a_\tau = a_w = 0$, $(e-1)b_\theta = b_\tau = b_w = 0$, $\Sigma_D = \varnothing$, the ratio θ/ϕ is constant, and $\mathcal{Y}_g = 0$.

Order-Preserving Maps Theorem

Let us write our problem in the form:

$$F(\phi) := \begin{pmatrix} -\Delta\phi + f(\phi) \\ \gamma_N \partial_\nu \phi + h(\phi) \\ \gamma_D \phi - \phi_D \end{pmatrix} = 0.$$

Say ψ is *super-solution* if $F(\psi) \geqslant 0$, and *sub-solution* if $F(\psi) \leqslant 0$, component-wise.

The following theorem from [HT13] extends the standard argument used for closed manifolds (cf. [Ise95, Max05a]) to manifolds with boundary; note that the required sub- and super-solutions need only satisfy inequalities in both the interior and on the boundary.

Theorem 24 (Order-Preserving Maps w/ Boundaries)

Suppose that the signs of the coefficients a_τ, a_w, b_θ, b_τ, b_w, and $b_H - \frac{n-2}{2}H$ are locally constant, and let $\phi_D > 0$. Let $\phi_-, \phi_+ \in W^{s,p}(M)$ be respectively sub- and super-solutions satisfying $0 < \phi_- \leqslant \phi_+$. Then there exists a positive solution $\phi \in [\phi_-, \phi_+]_{s,p}$ to the Lichnerowicz problem.

Existence Results - Defocusing Case

We start with metrics with nonnegative Yamabe invariant. In the following theorem from [HT13], the symbol $\vee$ denotes the logical disjunction (or logical OR).

Theorem 25 (Existence - Defocusing and $\mathcal{Y}_g \geqslant 0$)

Let $\mathcal{Y}_g \geqslant 0$. Let the coefficients of the Lichnerowicz problem satisfy $a_\tau \geqslant 0$, $a_w \geqslant 0$, $b_H \geqslant \frac{n-2}{2}H$, $(e-1)b_\theta \geqslant 0$ with $e \neq 1$, $b_\tau \geqslant 0$, $b_w \leqslant 0$, and $\phi_D > 0$. Then there exists a positive solution $\phi \in W^{s,p}(M)$ of the Lichnerowicz problem if and only if one of the following conditions holds:

a) $\Sigma_D \neq \varnothing$;

b) $\Sigma_D = \varnothing$, $b_\theta = 0$, $\big(\mathcal{Y}_g > 0 \vee a_\tau \neq 0 \vee b_H \neq \frac{n-2}{2}H \vee b_\tau \neq 0\big)$, and $(a_w \neq 0 \vee b_w \neq 0)$;

c) $\Sigma_D = \varnothing$, $b_\theta \neq 0$, $b_\theta \geqslant 0$, and $(a_w \neq 0 \vee b_w \neq 0)$;

d) $\Sigma_D = \varnothing$, $b_\theta \neq 0$, $b_\theta \leqslant 0$, and $\big(\mathcal{Y}_g > 0 \vee a_\tau \neq 0 \vee b_H \neq \frac{n-2}{2}H \vee b_\tau \neq 0\big)$;

e) $\Sigma_D = \varnothing$, $b_\theta = b_\tau = b_w = 0$, $b_H = \frac{n-2}{2}H$, $a_\tau = a_w = 0$, and $\mathcal{Y}_g = 0$.

Existence Results - Defocusing Case

The next theorem from [HT13] treats metrics with negative Yamabe invariant, and reduces the Lichnerowicz problem into a prescribed scalar curvature problem.

Theorem 26 (Existence - Defocusing and $\mathcal{Y}_g < 0$)

Let $\mathcal{Y}_g < 0$. Let the coefficients of the Lichnerowicz problem satisfy $a_\tau \geqslant 0$, $a_w \geqslant 0$, $b_H \leqslant \frac{n-2}{2} H$, $(e-1)b_\theta \geqslant 0$ with $e \neq 1$, $b_\tau \geqslant 0$, $b_w \leqslant 0$, and $\phi_D > 0$. Then there exists a positive solution $\phi \in W^{s,p}(M)$ of the Lichnerowicz problem if and only if there exists a positive solution $u \in W^{s,p}(M)$ to the following problem

$$\begin{aligned} -\Delta u + a_R u + a_\tau u^{2\bar{q}-1} &= 0, \\ \gamma_N \partial_\nu u + b_H u + b_\tau u^{\bar{q}} + b_\theta^+ u^e &= 0, \\ \gamma_D u &= 1, \end{aligned} \tag{35}$$

where $b_\theta^+ = \max\{0, b_\theta\}$.

There are also partial results in [HT13] for the non-defocusing case, but will not be outlined in this talk.

Non-CMC case: Main Results in [HMT14]

What about the non-CMC case?

In fact, even the CMC case was not yet discussed; this is because the CMC assumption does not actually decouple the constraints due to the boundary coupling, and we have only solved the Lichnerowicz equation.

The extension of the results in [HT13] to the non-CMC (far, near, and also CMC itself) is considered in [HMT14].

Some of the main results appearing in [HMT14] are:

- Number of necessary supporting results for momentum constraint that were not needed for pure Lichnerowicz case in [HT13].
- Construction of upper and lower barriers that respect trapped surface conditions in coupled setting (delicate boundary coupling).
- Combination of Schauder argument from [HNT09] with results for Lichnerowicz equation from [HT13] to give existence results for near-CMC and far-CMC data, analogous to known results for closed manifolds.
- CMC case comes as (still coupled) special case of near-CMC result.

Non-CMC case: The Setup

Assume compact domain $\mathcal{M}$ has boundary $\Sigma = \partial\mathcal{M} = \Sigma_I \cup \Sigma_E$, where boundary segments Σ_I and Σ_E decomposed further into finite segments:

$$\Sigma_I = \bigcup_{i=1}^{M} \Sigma_i, \quad \Sigma_E = \bigcup_{i=M+1}^{N} \Sigma_i, \quad (M < N), \quad \Sigma_i \cap \Sigma_j = \emptyset \quad \text{if} \quad i \neq j. \tag{36}$$

We consider the following system:

$$L\phi + a_R\phi + a_\tau\phi^5 - a_w\phi^7 - a_\rho\phi^{-3} = 0, \tag{37}$$

$$\mathbb{L}w + b_\tau\phi^6 + b_j = 0, \tag{38}$$

where L, $\mathbb{L}$, a_R, a_τ, a_w, a_ρ, b_τ, b_j as before, subject to boundary conditions:

$$\partial_\nu\phi + \frac{1}{2}H\phi + \left(\frac{1}{2}\tau - \frac{1}{4}\theta_-\right)\phi^3 - \frac{1}{4}S(\nu,\nu)\phi^{-3} = 0, \quad \text{on } \Sigma_I, \tag{39}$$

$$(\mathcal{L}w)^{ab}\nu_b = V^a, \quad \text{on } \Sigma_I, \tag{40}$$

$$\partial_\nu\phi + c\phi = g, \quad \text{on } \Sigma_E, \tag{41}$$

$$(\mathcal{L}w)^{ab}\nu_b + C^a_b w^b = 0, \quad \text{on } \Sigma_E, \tag{42}$$

where S, H, θ_- are traceless symmeric tensor, mean extrinsic boundary curvature, and incoming null geodesic expansion factor. In (39)-(42) we assume:

$$c > 0, \quad g > 0 \quad \text{and} \quad g = \delta(c + \mathcal{O}(R^{-3})), \quad \delta > 0, \tag{43}$$

$$\exists\alpha > 0 \quad \text{such that} \quad \int_{\partial\mathcal{M}} C_{ab}V^aV^b \geqslant \alpha|V|_{L^2(\partial\mathcal{M})}, \quad \forall V \in L^2.$$

Theorem 27 (Near-CMC and CMC $W^{s,p}$ Solutions)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, compact Riemannian manifold with boundary satisfying (36). Let $h_{ab} \in W^{s,p}(T^0_2\mathcal{M})$, with $p \in (1,\infty)$, $s \in (1+\frac{3}{p},\infty)$ given. With $d = s - \frac{3}{p}$, select q and e so:

- $\frac{1}{q} \in (0,1) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}] \cap [\frac{1-d}{3}, \frac{3+sp}{6p})$,
- $e \in [1,\infty) \cap [s-1, s] \cap [\frac{3}{q}+d-1, \frac{3}{q}+d]$.

Let (43) hold and assume the data satisfies:

- $\theta_- \in W^{s-1-\frac{1}{p},p}(\Sigma_I)$, $\quad c, g \in W^{s-1-\frac{1}{p},p}(\Sigma_E)$, $\quad C^a_b \in W^{e-1-\frac{1}{q},q}(T^1_1\Sigma_E)$,
- $V \in W^{e-1,q}$, $\quad V^a\nu_a = (2\tau + |\theta_-|/2)B^6 - \sigma(\nu,\nu)$,
- $\tau \in W^{s-1,p}$ *if* $e \geqslant 2$, *and* $\tau \in W^{1,z} \cap L^\infty$ *otherwise, with* $z = \frac{3p}{3+\max\{0,2-s\}p}$,
- $(4\tau^\vee + |\theta|^\vee) > 0$ *on* Σ_I,
- $\sigma \in W^{e-1,q}$, $\quad \rho \in W^{s-2,p}_+$, $\quad j \in W^{e-2,q}$.

In addition, assume $a^\vee_\tau > \theta k_1$ (the near-CMC condition), where $\theta k_1 = 2C^2(\|b_\tau\|_z)^2$, with C is a positive constant. If at least one of the following hold:

(a) $\rho^\vee > 0$,

(b) $a^\vee_\sigma$ *is sufficiently large,*

then there exists a solution $\phi \in W^{s,p}$ with $\phi > 0$ and $w \in W^{e,q}$ to (37)–(42). Moreover, with an additional smallness assumption on τ on Σ_I, the marginally trapped surface boundary condition is satisfied.

Theorem 28 (Non-CMC $W^{s,p}$ solutions)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, compact Riemannian manifold with boundary satisfying (36). Let $h_{ab} \in W^{s,p}(T^0_2\mathcal{M})$ and be in $\mathcal{Y}^+$, $p \in (1,\infty)$, $s \in (1+\frac{3}{p},\infty)$ given. With $d = s - \frac{3}{p}$, select q and e to satisfy:

- $\frac{1}{q} \in (0,1) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}] \cap [\frac{1-d}{3}, \frac{3+sp}{6p})$,
- $e \in [1,\infty) \cap [s-1, s] \cap [\frac{3}{q}+d-1, \frac{3}{q}+d]$.

Let (43) hold and assume the data satisfies:

- $\theta_- \in W^{s-1-\frac{1}{p},p}(\Sigma_I) \cap L^\infty(\Sigma_I)$, $\quad c, g \in W^{s-1-\frac{1}{p},p}(\Sigma_E)$, $\quad C^a_b \in W^{e-1-\frac{1}{q},q}(T^1_1\Sigma_E)$,
- $V \in W^{e-1,q}$, $\quad V^a\nu_a = (2\tau + |\theta_-|/2)B^6 - \sigma(\nu,\nu)$,
- $\tau \in W^{s-1,p}$ *if* $s \geqslant 2$, *and* $\tau \in W^{1,z} \cap L^\infty$ *otherwise, with* $z = \frac{3p}{3+\max\{0,2-s\}p}$,
- $(4\tau^\vee + |\theta|^\vee) > 0$ *on* Σ_I,
- $\sigma \in W^{e-1,q}$, $\quad \rho \in W^{s-2,p}_+ \cap L^\infty \setminus \{0\}$, $\quad j \in W^{e-2,q}$, *sufficiently small.*

Additionally assume that at least one of the following hold:

(a) $\delta > 0$ *is sufficiently small in (43);*

(b) $a^\vee_R > 0$ *is sufficiently large;*

(c) $\|\theta_-\|_\infty$ *is sufficiently small, and* $D\tau$ *is sufficiently small.*

Then:

Case (a): The function B can be chosen so the marginally trapped surface condition is satisfied, and subsequently there exists a solution $\phi \in W^{s,p}$ with $\phi > 0$ and $w \in W^{e,q}$ to equations (37)–(42).

Cases (b) and (c): There exists a solution $\phi \in W^{s,p}$ with $\phi > 0$ and $w \in W^{e,q}$ to (37)–(42). With an additional smallness assumption on τ on Σ_I, the marginally trapped surface condition may be satisfied.

Asymptotically Euclidean Case

[DIM14] J. Dilts, J. Isenberg, R. Mazzeo, and C. Meier. Non-cmc solutions of the Einstein constraint equations on asymptotically euclidean manifolds. 2014 Class. Quantum Grav. **31** (2014), 065001.

[HMa14] MH and C. Meier, *Non-CMC solutions of the Einstein constraint equations on asymptotically Euclidean manifolds with apparent horizon boundaries*, Class. Quantum Grav., 32 (2014), No. 2, pp. 1-25. Available as arXiv:1403.4549 [gr-qc].

[BH14] A. Behezadan and MH, *Rough solutions of the Einstein constraint equations on asymptotically flat manifolds without near-CMC conditions*, Preprint. Available as arXiv:1504.04661 [gr-qc].

The most complete mathematical model of general relativity involves the evolution and constraint equations on open, asymptotically Euclidean manifolds, with black hole interior boundary conditions.

Existence results analogues to those for closed manifolds have been known since shortly after the closed results developed.

In [HMa14], we develop non-CMC existence results for asymptotically Euclidean manifolds with black hole interior boundaries.

In [BH14], we extend this work to rough metrics, in some sense completing part of the research program begun in 2004/2005.

AE Case: Main Theorems (Smooth metric)

Theorem 29 (Non-CMC)

Suppose that $(\mathcal{M}, g)$ is asymptotically Euclidean of class $W^{2,p}_\gamma$ with $p > n$ and $2 - n < \gamma < 0$. Assume that $2 - n < \delta < \gamma/2$, and the data satisfies:

- $g \in \mathcal{Y}^+$,
- $\tau \in W^{1,p}_{\delta-1}$,
- $\sigma \in W^{1,2p}_{\delta-1}$ *with* $\|\sigma\|_{L^\infty_{\delta-1}}$ *sufficiently small,*
- $\rho \in L^\infty_{\gamma-2}$ *with* $\|\rho\|_{L^\infty_{\delta-2}}$ *sufficiently small,*
- $J \in L^p_{\delta-2}$ *with* $\|J\|_{L^p_{\delta-2}}$ *sufficiently small,*
- $\theta_- \in W^{1-\frac{1}{p},p}(\Sigma)$, $\quad \theta_- < 0$,
- $V \in W^{1,p}$, $\quad V|_\Sigma = \left(((n-1)\tau + |\theta_-|/2)\psi^N - \sigma(\nu,\nu)\right)\nu$,
- $((n-1)\tau + |\theta_-|/2) > 0$ *and* $\|(n-1)\tau + |\theta_-|/2\|_{W^{1-\frac{1}{p},p}(\Sigma)}$ *sufficiently small.*

Then on each end E_i there exists an interval $\mathcal{I}_i \subset (0,\infty)$ such that if $A_i \in \mathcal{I}_i$ are freely specified constants and ω is the associated harmonic function, there exists a solution (ϕ, W) to the conformal equations with boundary conditions (39)–(40) such that $\phi - \omega \in W^{2,p}_\gamma$ and $W \in W^{2,p}_\delta$. Moreover, the function ψ can be chosen so that (ϕ, W) satisfies the marginally trapped surface boundary conditions.

Theorem 30 (Near-CMC with bounded R and H)

Suppose that $(\mathcal{M}, g)$ is asymptotically Euclidean of class $W^{2,p}_\gamma$ with $p > n$ and $2 - n < \gamma < 0$. Assume that $2 - n < \delta < \gamma/2$, and the data satisfies:

- $\|\nabla\tau\|_{L^p_{\delta-2}}$ *is sufficiently small,*
- $\sigma \in W^{1,2p}_{\gamma-1}$,
- $\rho \in L^p_{\gamma-2}$,
- $J \in L^{p_{\delta-2}}$,
- $\theta_- \in W^{1-\frac{1}{p},p}(\Sigma), \quad \theta_- < 0$,
- $bV \in W^{1,p}, \quad V|_\Sigma = \left(((n-1)\tau + |\theta_-|/2)\psi^N - \sigma(\nu,\nu)\right)\nu$,
- $(2(n-1)\tau + |\theta_-|) > 0$ *is sufficiently small on* Σ.

Let $A_i \in [1, \infty)$ be freely specified constants on each end E_i and let ω be the associated harmonic function. Then if

- $-c_n R \leqslant b_n \tau^2$ *on* $\{x \in \mathcal{M} : R(x) < 0\}$,
- $-H \leqslant (\tau + |\theta_-|/(n-1))$ *on* $\{x \in \Sigma : H(x) < 0\}$,

there exists a solution (ϕ, W) to the conformal equations with boundary conditions (39)-(40) such that $\phi - \omega \in W^{2,p}_\gamma$ and $W \in W^{2,p}_\delta$. Moreover, the function ψ can be chosen so that (ϕ, W) satisfies the marginally trapped surface boundary conditions.

AE Case: Main Theorems (Smooth metric)

Theorem 31 (Near-CMC with $g \in \mathcal{Y}^+$)

Suppose that $(\mathcal{M}, g)$ is asymptotically Euclidean of class $W^{2,p}_\gamma$ with $p > n$ and $2 - n < \gamma < 0$. Assume that $2 - n < \delta < \gamma/2$, and the data satisfies:

- $g \in \mathcal{Y}^+$,
- $\|\tau\|_{W^{1,p}_{\delta-1}}$ *is sufficiently small, and* $\tau \geqslant 0$ *on* Σ,
- $\sigma \in W^{1,2p}_{\gamma-1}$,
- $\rho \in L^p_{\gamma-2}$,
- $\|J\|_{L^p_{\delta-2}}$ *is sufficiently small,*
- $\theta_- = 0$,
- $V \in W^{1,p}$, $\quad V|_\Sigma = \left(\left((n-1)\tau\right)\phi^N - \sigma(\nu,\nu)\right)\nu$.

Then if $A_i \in (0, \infty)$ are freely specified constants on each end E_i and ω is the associated harmonic function, there exists a unique solution (ϕ, W) to the conformal equations with marginally trapped surface boundary conditions such that $\phi - \omega \in W^{2,p}_\gamma$ and $W \in W^{2,p}_\delta$.

Note: The proof of this near-CMC result goes through the Implicit Function Theorem, which subsequently gives both existence and uniqueness in this case.

AE Case: Main Theorem (Rough metric)

Theorem 32 (Non-CMC Rough AE Solutions)

Let (M, h) be a 3-dimensional AE Riemannian manifold of class $W^{s,p}_{\delta}$ where $p \in (1, \infty)$, $s \in (1 + \frac{3}{p}, \infty)$ and $-1 < \delta < 0$ are given. Suppose h admits no nontrivial conformal Killing field and is in the positive Yamabe class. Let $\beta \in (-1, \delta]$. Select q and e to satisfy:

- $\frac{1}{q} \in (0, 1) \cap (0, \frac{s-1}{3}) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}]$,
- $e \in (1 + \frac{3}{q}, \infty) \cap [s-1, s] \cap [\frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p}]$.

Let $q = p$ if $e = s \notin \mathbb{N}_0$. Moreover if $s > 2$, $s \notin \mathbb{N}_0$, assume $e < s$. Assume that the data satisfies:

- $\tau \in W^{e-1,q}_{\beta-1}$ *if* $e \geq 2$ *and* $\tau \in W^{1,z}_{\beta-1}$ *otherwise, where* $z = \dfrac{3q}{3 + (2-e)q}$
- $\sigma \in W^{e-1,q}_{\beta-1}$,
- $J \in \mathbf{W}^{e-2,q}_{\beta-2}$, $\rho \in W^{s-2,p}_{\beta-2} \cap L^{\infty}_{2\beta-2}$, $\rho \geq 0$ *(ρ can be identically zero).*

If $\mu > 0$ is chosen to be sufficiently small and if $\|\sigma\|_{L^{\infty}_{\beta-1}}$, $\|\rho\|_{L^{\infty}_{2\beta-2}}$, and $\|J\|_{\mathbf{W}^{e-2,q}_{\beta-2}}$ are sufficiently small, then there exists $\phi = \psi + \mu > 0$, $\psi \in W^{s,p}_{\delta}$ and $W \in \mathbf{W}^{e,q}_{\beta}$ solving the constraints.

Multiplicity Results

[HK11] MH and V. Kungurtsev, *Numerical bifurcation analysis of conformal formulations of the Einstein constraints*, Phys. Rev. D, 84 (2011), pp. 124038(1)–124038(8). Available as arXiv:1107.0262 [math.NA].

[HM14] MH and C. Meier, *Non-uniqueness of solutions to the conformal formulation*, Accepted for publication. Available as arXiv:1210.2156 [gr-qc].

One of the features of the new non-CMC existence results in [HNT08, HNT09] (and similar results) is lack of uniqueness.

In 2009, Maxwell explicitly demonstrated existence of multiple solutions for a special symmetric model [Max09b], and now also [Max14b].

Folds in solution curves observed numerically by Pfeiffer, O'Murchadha and others for non-standard formulations of the constraints; i.e., the mechanism is different from Maxwell results.

The non-standard formulation giving rise to this behavior has un-scaled matter sources:

$$L\phi + a_R\phi + a_\tau\phi^5 - a_w\phi^{-7} - a_\rho\phi^5 = 0, \tag{44}$$

$$\mathbb{L}w + b_\tau^b\phi^6 + b_j^b\phi^{10} = 0, \tag{45}$$

where as before, $L\phi = -\Delta\phi$ and $(\mathbb{L}w)^a = -\nabla_b(\mathcal{L}w)^{ab}$.

Multiplicity Results

We wanted to examine more carefully (both numerically and mathematically) the folds that had been observed numerically in this non-standard formulation using somewhat ad-hoc methods.

To this end, in [HK11], we applied pseudo-arclength numerical continuation to numerically track the parameterized solution curve in the problem previously examined by Pfeiffer and O'Murchadha, and numerically identify a fold.

In [HM14], using tools from analytic bifurcation theory, we show the linearization of the non-standard constraint system develops a one-dimensional kernel in both the CMC and Non-CMC cases.

Through Liapunov-Schmidt reduction, we show solutions with unscaled data are non-unique by determining an explicit solution curve, and analyze its behavior in the neighborhood of a particular solution.

The technique involves the following λ-parameterization of the model:

$$L\phi + a_R\phi + \lambda^2 a_\tau \phi^5 - a_w\phi^{-7} - e^{-\lambda} a_\rho \phi^5 = 0, \tag{46}$$

$$\mathbb{L}w + \lambda b_\tau^b \phi^6 = 0, \tag{47}$$

Multiplicity Results

To explain the main results, define $a_\sigma = \frac{1}{8}\sigma^2$, and

$$G(\phi, \lambda) = L\phi + a_R\phi + \lambda^2 a_\tau \phi^5 - a_\sigma \phi^{-7} - e^{-\lambda} a_\rho \phi^5, \tag{48}$$

$$F((\phi, w), \lambda) = \begin{bmatrix} L\phi + a_R\phi + \lambda^2 a_\tau \phi^5 - a_w \phi^{-7} - e^{-\lambda} a_\rho \phi^5 \\ \mathbb{L}w + \lambda b_\tau^b \phi^6 \end{bmatrix}. \tag{49}$$

Theorem 33 (CMC: Critical Values)

Let $D_\phi G(\phi, \lambda)$ be the Fréchet derivative of (48) with respect to ϕ. Then:

1. *There exists a critical value of $\rho = \rho_c$ and a constant ϕ_c such that when $\rho = \rho_c$, $G(\phi, \lambda) = 0$ has a solution if and only if $\lambda \geq 0$.*
2. *Furthermore, $\dim \ker(D_\phi G(\phi_c, 0)) = 1$ and it is spanned by the constant function $\phi = 1$.*
3. *Moreover, we can determine explicit values of ρ_c and ϕ_c:*

$$\rho_c = \frac{R^{\frac{3}{2}}}{24\sqrt{3}\pi|\sigma|}, \qquad \phi_c = \left(\frac{R}{24\pi\rho_c}\right)^{\frac{1}{4}}. \tag{50}$$

Multiplicity Results

Theorem 34 (CMC: Multiplicity)

Suppose τ is constant, so that our problem is $G(\phi, \lambda) = 0$. Let $\rho = \rho_c$, with ρ_c as in Theorem 33. Then:

1. *There exists a neighborhood of $(\phi_c, 0)$ such that all solutions to $G(\phi, \lambda) = 0$ in this neighborhood lie on a smooth solution curve $\{\phi(s), \lambda(s)\}$ that has the form*

$$\phi(s) = \phi_c + s + O(s^2),$$
$$\lambda(s) = \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3), \quad (\ddot{\lambda}(0) \neq 0).$$

2. *In particular, there exists a $\delta > 0$ such that for all $0 < \lambda < \delta$ there exist at least two distinct solutions $\phi_{1,\lambda} \neq \phi_{2,\lambda}$ to $G(\phi, \lambda) = 0$.*

Theorem 35 (Non-CMC: 1-dimensional nullspace)

Let $D_X F((\phi, w), \lambda)$ be the Fréchet derivative of (49) w.r.t. $X = (\phi, w)$. Let ρ_c, ϕ_c be as in Thm 33. If $\rho = \rho_c$, then $\dim \ker(D_X F((\phi_c, 0), 0)) = 1$ and $\ker(D_X F((\phi_c, 0), 0))$ is spanned by constant vector $[1, 0]^T$.

Theorem 36 (Non-CMC: Multiplicity)

Suppose $\tau \in C^{1,\alpha}(\mathcal{M})$ is non-constant and let $F((\phi, w), \lambda)$ be defined as in (49), so that our problem is: $F((\phi, w), \lambda) = 0$. Let ρ_c and ϕ_c be defined as in Theorem 33. If $\rho = \rho_c$, then:

1. *There exists a neighborhood B of $((\phi_c, w), 0)$ such that all solutions to $F((\phi, w), \lambda) = 0$ in B lie on a smooth curve of the form*

$$\phi(s) = \phi_c + s + \frac{1}{2}\ddot{\lambda}(0)u(x)s^2 + O(s^3),$$
$$w(s) = \frac{1}{2}\ddot{\lambda}(0)v(x)s^2 + O(s^3),$$
$$\lambda(s) = \frac{1}{2}\ddot{\lambda}(0)s^2 + O(s^3), \quad (\ddot{\lambda}(0) \neq 0),$$

   where $u(x) \in C^{2,\alpha}(\mathcal{M})$, $v(x) \in C^{2,\alpha}(\mathcal{TM})$ and $v(x) \neq 0$.

2. *In particular, there exists a $\delta > 0$ s.t. for all $0 < \lambda < \delta$ there exist elements $(\phi_{1,\lambda}, w_{1,\lambda}), (\phi_{2,\lambda}, w_{2,\lambda}) \in C^{2,\alpha}(\mathcal{M}) \oplus C^{2,\alpha}(\mathcal{TM})$ s.t.*

$$F((\phi_{i,\lambda}, w_{i,\lambda}), \lambda) = 0, \text{ for } i \in \{1, 2\}, \text{ and } (\phi_{1,\lambda}, w_{1,\lambda}) \neq (\phi_{2,\lambda}, w_{2,\lambda}).$$

Thank You for Listening!

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Results from Our Group

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