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§1.2 Taylor's Theorem

Recall from calculus the Taylor's series for a function, $f(x)$, expanded about some number, c, is written as

$$f(x) \sim a_0 + a_1 (x - c) + a_2 (x - c)^2 + \ldots.$$

Here the symbol $\sim$ is used to denote a "formal series," meaning that convergence is not guaranteed in general. The constants a_i are related to the function f and its derivatives evaluated at c. When $c = 0$, this is a MacLaurin series.

For example we have the following Taylor's series (with $c = 0$):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad (1)$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \qquad (2)$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \qquad (3)$$

Theorem 1 (Taylor's Theorem). If $f(x)$ has derivatives of order $0, 1, 2, \ldots, n+1$ on the closed interval $[a, b]$, then for any x and c in this interval

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)\ (x-c)^k}{k!} + \frac{f^{(n+1)}(\xi)\ (x-c)^{n+1}}{(n+1)!},$$

where ξ is some number between x and c.

We will use this theorem again and again in this class. The main usage is to approximate a function by the first few terms of its Taylor's series expansion; the theorem then tells us the approximation is "as good" as the final term, also known as the *error term*. That is, we can make the following manipulation:

$$\begin{aligned} f(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(c)\ (x-c)^k}{k!} + \frac{f^{(n+1)}(\xi)\ (x-c)^{n+1}}{(n+1)!} \\ f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(c)\ (x-c)^k}{k!} &= \frac{f^{(n+1)}(\xi)\ (x-c)^{n+1}}{(n+1)!} \\ \left| f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(c)\ (x-c)^k}{k!} \right| &= \frac{\left|f^{(n+1)}(\xi)\right| |x-c|^{n+1}}{(n+1)!}. \end{aligned}$$

On the left hand side is the difference between $f(x)$ and its approximation by Taylor's series. We will then use our knowledge about $f^{(n+1)}(\xi)$ on the interval $[a, b]$ to find some constant M such that

$$\left| f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(c)\ (x-c)^k}{k!} \right| = \frac{\left| f^{(n+1)}(\xi) \right| |x-c|^{n+1}}{(n+1)!} \leq M\, |x-c|^{n+1}.$$

Example 2. Find an approximation for $f(x) = \sin x$, expanded about $c = 0$, using $n = 3$.
Solution: Solving for $f^{(k)}$ is fairly easy for this function. We find that

$$\begin{aligned} f(x) = \sin x &= \sin(0) + \frac{\cos(0)\, x}{1!} + \frac{-\sin(0)\, x^2}{2!} + \frac{-\cos(0)\, x^3}{3!} + \frac{\sin(\xi)\, x^4}{4!} \\ &= x - \frac{x^3}{6} + \frac{\sin(\xi)\, x^4}{24}, \end{aligned}$$

so

$$\left| \sin x - \left(x - \frac{x^3}{6} \right) \right| = \left| \frac{\sin(\xi)\, x^4}{24} \right| \leq \frac{x^4}{24},$$

because $|\sin(\xi)| \leq 1$.

Example 3. Apply Taylor's Theorem for the case $n = 1$. In this case the theorem tells us the following:
Given a function, $f(x)$ with a continuous derivative on $[a, b]$, then

$$f(x) = f(c) + f'(\xi)(x - c)$$

for some ξ between x, c when x, c are in $[a, b]$.
This is the Mean Value Theorem. As a one-liner, the MVT says that at some time during a trip, your velocity is the same as your average velocity for the trip.