

§1.2 Taylor's Theorem

Recall Taylor's theorem; we will be using it fairly extensively in this class.

Example 1. Apply Taylor's Theorem to expand $f(x) = x^3 - 21x^2 + 17$ around $c = 1$.

Solution: Simple calculus gives us

$$\begin{aligned}f^{(0)}(x) &= x^3 - 21x^2 + 17, \\f^{(1)}(x) &= 3x^2 - 42x, \\f^{(2)}(x) &= 6x - 42, \\f^{(3)}(x) &= 6, \\f^{(k)}(x) &= 0.\end{aligned}$$

with the last holding for $k > 3$. Evaluating these at $c = 1$ gives

$$f(x) = -3 + -39(x-1) + \frac{-36(x-1)^2}{2} + \frac{6(x-1)^3}{6}.$$

Note there is no error term, since the higher order derivatives are identically zero. By carrying out simple algebra, you will find that the above expansion is in fact the function $f(x)$.

There is an alternative form of Taylor's Theorem, in this case substituting $x+h$ for x , and x for c in the more general version. This gives

Theorem 2 (Taylor's Theorem, Alternative Form). If $f(x)$ has derivatives of order $0, 1, 2, \dots, n+1$ on the closed interval $[a, b]$, then for any x in this interval and any h such that $x+h$ is in this interval,

$$f(x+h) = \sum_{k=0}^n \frac{f^{(k)}(x) (h)^k}{k!} + \frac{f^{(n+1)}(\xi) (h)^{n+1}}{(n+1)!},$$

where ξ is some number between x and $x+h$.

We generally apply this form of the theorem with $h \rightarrow 0$. This leads to a discussion on the matter of *Orders of Convergence*. In this class, we will say that some quantity, Q is "Big-O" of some power of h , say h^k if there is some constant M such that

$$Q \leq Mh^k.$$

In this case we sometimes write

$$Q = \mathcal{O}(h^k),$$

although the meaning of the equality is vague. Some people insist that “Big-O” is a class of functions with this convergence property, and a quantity can be a member of this class.

Some interesting things to think about “Big-O” functions: if $Q = \mathcal{O}(h^k)$, and $T = \mathcal{O}(h^m)$ then what is $Q + T$? How about QT ? If $Q = \mathcal{O}(h^k)$ isn't it the case that $Q = \mathcal{O}(h^m)$ for $k < m$?

Roughly speaking, through use of the “Big-O” function we can write an expression without “sweating the small stuff.” This can give us an intuitive understanding of how an approximation works, without losing too many of the details.

Example 3. To make this notion clear, we look at the expansion of $\ln x$.

$$\ln(x+h) = \ln x + \frac{(1/x)h}{1} + \frac{(-1/x^2)h^2}{2} + \frac{(2/\xi^3)h^3}{6}$$

Letting $x = 1$, we have

$$\ln(1+h) = h - \frac{h^2}{2} + \frac{1}{3\xi^3}h^3.$$

Using the fact that ξ is between 1 and $1+h$, as long as h is relatively small (say smaller than $\frac{1}{2}$), the term $\frac{1}{3\xi^3}$ can be bounded by a constant, and thus

$$\ln(1+h) = h - \frac{h^2}{2} + \mathcal{O}(h^3).$$

Another theorem for proving convergence is the following:

Theorem 4 (Alternating Series Theorem). If $a_0 \geq a_1 \geq a_2 \geq a_3 \geq \dots \geq 0$, and $\lim_{k \rightarrow \infty} a_k = 0$ then the series

$$\sum_{k=0}^{\infty} (-1)^k a_k = a_0 - a_1 + a_2 - \dots$$

converges. Moreover, if you let S be the value of the sum, and S_n be the partial sum $\sum_{k=0}^n (-1)^k a_k$, then

$$|S_n - S| \leq a_{n+1}.$$

This theorem can help us figure out how many terms to take in an approximation.

Example 5. Use Taylor's Theorem on $f(x) = \ln(1+x)$, then use the AST to figure out how many terms are needed to get an accurate approximation for $\ln 2$.

Solution: The expansion for this function is actually

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}.$$

Thus

$$\ln 2 = \ln(1+x) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

This is an alternating series with terms $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$. How many terms are sufficient? Suppose we wish to find an approximation of $\ln 2$ good to 9 decimal places. Thus we want an error smaller than $\frac{1}{2} \times 10^{-9}$. The AST tells us that

$$|S_n - n| \leq a_{n+1} = \frac{1}{n+1}.$$

Thus it suffices to take $\frac{1}{n+1} \leq \frac{1}{2} \times 10^{-9}$. That is, the AST tells us we need to take 2 billion terms in our approximation; this is poor.

§2.3 Loss of Significance

Generally speaking, a computer stores a number x as a mantissa and exponent, that is $x = \pm r \times 10^k$, where r is a rational number of a given number of digits in $[0.1, 1)$, and k is an integer in a certain range.

The number of significant digits in r is usually determined by the user's input. Operations on numbers stored in this way follow a "lowest common denominator" type of rule, *i.e.*, precision cannot be gained but can be lost. Thus for example if you add the two quantities 0.171717 and 0.51, then the result should only have two significant digits; the precision of the first measurement is lost in the uncertainty of the second.

This is as it should be. However, a loss of significance can be incurred if two nearly equal quantities are subtracted from one another. Thus if I were to direct my computer to subtract 0.177241 from 0.177589, the result would be $.348 \times 10^{-3}$, and three significant digits have been lost. This kind of loss is often avoidable by rewriting the expression to avoid this so-called *subtractive cancellation*.

Example 6. Rewrite $\sqrt{x+1} - 1$ to be stable when x is near 0.

Solution: We *rationalize* the expression

$$\sqrt{x+1} - 1 = \sqrt{x+1} - 1 \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1} = \frac{x+1-1}{\sqrt{x+1}+1} = \frac{x}{\sqrt{x+1}+1}.$$

This expression has no subtractions, and so is not subject to subtractive cancelling.

Example 7. Rewrite $e^x - \cos x$ to be stable when x is near 0.

Solution: Look at the Taylor's Series expansion for these functions:

$$\begin{aligned} e^x - \cos x &= \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right] - \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right] \\ &= x + x^2 + \frac{x^3}{3!} + \mathcal{O}(x^4) \end{aligned}$$

This expression has no subtractions, and so is not subject to subtractive cancelling.