Steven Pav SPAV@MATH.UCSD.EDU

§3.3 Secant Method

The secant method for root finding is roughly based on Newton's method; however, it is not assumed that the derivative of the function is known, rather the derivative is approximated by the value of the function at some of the iterates, x_k. More specifically, the slope of the *tangent* line at $(x_k, f(x_k))$, which is $f'(x_k)$ is approximated by the slope of the *secant* line passing through $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$, which is

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

Thus the iterate x_{k+1} is the root of this secant line. That is, it is a root to the equation

$$y - f(x_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} (x - x_k) .$$

Since the root has a y value of 0, we have

$$-f(x_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} (x_{k+1} - x_k) , \qquad (1)$$

$$-\left(\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}\right) f(x_k) = x_{k+1} - x_k, \qquad (2)$$

$$x_k - \left(\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}\right) f(x_k) = x_{k+1}. \qquad (3)$$

$$(4)$$

You will note this is the recurrence of Newton's method, but with the slope $f'(x_k)$ substituted by the slope of the secant line. Note also that the secant method requires two initial guesses, x_0, x_1, but can be used on a black box function. The secant method can suffer from some of the same problems that Newton's method does, as we will see.

An iteration of the secant method is shown in Figure 1, along with the secant line.

Example 1. Consider the secant method used on $x^3 + x^2 - x - 1$, with $x_0 = 2, x_1 = \frac{1}{2}$.

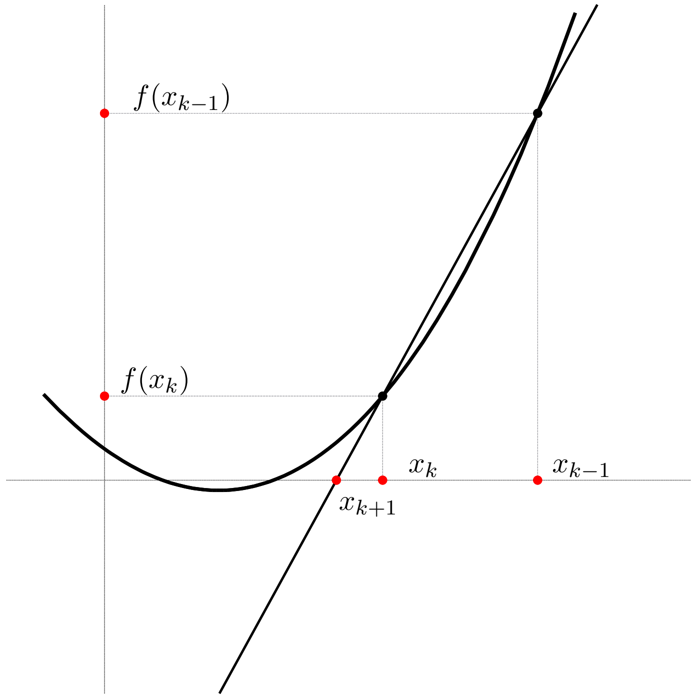

Figure 1: One iteration of the Secant method is shown for some quadratic function $f(x)$. The secant line through $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$ is shown. It happens to be the case that $|f(x_{k+1})|$ is smaller than $|f(x_k)|$, *i.e.*, x_{k+1} is a better guess than x_k.

Note that this function is continuous and has roots ± 1. We give the iterates here:

k	x_k	$f(x_k)$
0	2	9
1	0.5	-1.125
2	0.666666666666667	-0.925925925925926
3	1.44186046511628	2.63467367653163
4	0.868254072087394	-0.459842466254495
5	0.953491494113659	-0.177482458876898
6	1.00706900811804	0.0284762692197613
7	0.999661272951803	-0.0013544492875992
8	0.999997617569723	$-9.52969840528617 \times 10^{-06}$
9	1.0000000008072	$3.22880033820638 \times 10^{-09}$
10	0.999999999999998	$-7.43849426498855 \times 10^{-15}$

Problems As with Newton's method, convergence is dependant on $f(x)$, and the initial estimates x_0, x_1. We illustrate a few possible problems:

Example 2. Consider the secant method for the function $f(x) = \frac{\ln x}{x}$, with $x_0 = 3, x_1 = 4$. As with Newton's method, the iterates diverge towards infinity, looking for a nonexistant root. We give some iterates here:

k	x_k	$f(x_k)$
0	3	0.366204096222703
1	4	0.346573590279973
2	21.6548475770851	0.142011128224341
3	33.9111765137635	0.103911011441661
4	67.3380435135758	0.0625163004418104
5	117.820919458675	0.0404780904944712
6	210.543986613847	0.0254089165873003
7	366.889164762149	0.0160949419231219
8	637.060241341843	0.010135406045582
9	1096.54125113444	0.00638363233543847
10	1878.34688714646	0.00401318169994875
11	3201.94672271613	0.0025208146648422
12	5437.69020766155	0.00158175793894727
13	9203.60222260594	0.000991714984152597
14	15533.1606791089	0.000621298692241343
15	26149.7196085218	0.000388975250950428
16	43924.8466075548	0.000243375589137882
17	73636.673898472	0.000152191807070607

Example 3. Consider Newton's method applied to the function $f(x) = \frac{1}{1+x^2} - \frac{1}{17}$, with initial estimates $x_0 = -1, x_1 = 1$.
You can easily verify that $f(x_0) = f(x_1)$, and thus the secant line is horizontal. And thus x_2 is not defined.

Convergence We consider convergence analysis as in Newton's method. We assume that r is a root of $f(x)$, and let $e_n = r - x_n$. Because the secant method involves two iterates, we assume that we will find some relation between e_{k+1} and the previous two errors, e_k, e_{k-1}. We start with

$$\begin{aligned}
e_{k+1} &= r - x_{k+1} \\
&= r - \left[x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k)\right], \\
&= \frac{rf(x_k) - rf(x_{k-1})}{f(x_k) - f(x_{k-1})} - \frac{f(x_k)x_{k-1} - f(x_{k-1})x_k}{f(x_k) - f(x_{k-1})}, \\
&= \frac{f(x_k)e_{k-1} - f(x_{k-1})e_k}{f(x_k) - f(x_{k-1})}, \\
&= \frac{f(x_k)/e_k - f(x_{k-1})/e_{k-1}}{f(x_k) - f(x_{k-1})} e_k e_{k-1}, \\
&= \left[\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}\right] \left[\frac{f(x_k)/e_k - f(x_{k-1})/e_{k-1}}{x_k - x_{k-1}}\right] e_k e_{k-1}. \qquad (5)
\end{aligned}$$

Now recall Taylor's theorem, expanding about r:

$$\begin{aligned}
f(x_k) = f(r + x_k - r) = f(r - e_k) &= f(r) + f'(r)(-e_k) + f''(r)\frac{e_k^2}{2} - f'''(\xi_k)\frac{e_k^3}{6}, \\
&= f'(r)(-e_k) + f''(r)\frac{e_k^2}{2} - f'''(\xi_k)\frac{e_k^3}{6}, \\
f(x_k)/e_k &= -f'(r) + f''(r)\frac{e_k}{2} - f'''(\xi_k)\frac{e_k^2}{6}, \\
&= -f'(r) + f''(r)\frac{e_k}{2} + \mathcal{O}\left(e_k^2\right).
\end{aligned}$$

where ξ_k is between x_k and $r = x_k + e_k$.

Line up this equation for $k, k-1$, then subtract to get

$$f(x_k)/e_k - f(x_{k-1})/e_{k-1} = f''(r)\frac{e_k - e_{k-1}}{2} + \mathcal{O}\left(e_k^2\right) + \mathcal{O}\left(e_{k-1}^2\right).$$

Note that $x_k - x_{k-1} = x_k - r + r - x_{k-1} = e_{k-1} - e_k$, so

$$f(x_k)/e_k - f(x_{k-1})/e_{k-1} = -\frac{f''(r)}{2}\left[x_k - x_{k-1}\right] + \mathcal{O}\left(e_k^2\right) + \mathcal{O}\left(e_{k-1}^2\right).$$

Returning to equation 5, we get

$$e_{k+1} \approx \left[\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}\right]\left[-\frac{f''(r)}{2}\right] e_k e_{k-1}.$$

By Taylor's theorem, the first term is very near $\frac{1}{f'(r)}$, so

$$e_{k+1} \approx -\frac{f''(r)}{2f'(r)} e_k e_{k-1} = C e_k e_{k-1}.$$

We now postulate that the error terms for the secant method follow some power law of the following type:

$$\left|e_{k+1}\right| \sim A \left|e_k\right|^{\alpha}.$$

Recall that this held true for Newton's method, with $\alpha = 2$. We try to find the α for the secant method. Note that

$$\left|e_k\right| = A \left|e_{k-1}\right|^{\alpha},$$

so

$$\left|e_{k-1}\right| = A^{-\frac{1}{\alpha}} \left|e_k\right|^{\frac{1}{\alpha}}.$$

Then we have

$$A \left|e_k\right|^{\alpha} = \left|e_{k+1}\right| = C \left|e_k\right| \left|e_{k-1}\right| = C A^{-\frac{1}{\alpha}} \left|e_k\right|^{1+\frac{1}{\alpha}},$$

Since this equation is to hold for all e_k, we must have

$$\alpha = 1 + \frac{1}{\alpha}.$$

This is solved by $\alpha = \frac{1}{2}\left(1 + \sqrt{5}\right) \approx 1.62$. Thus we say that the secant method enjoys *superlinear* convergence; This is somewhere between the convergence rates for bisection and Newton's method.