2003/10/17	**M174 : Numerical Methods in Science and Engineering**
Steven Pav	SPAV@MATH.UCSD.EDU

§5.1 the Definite Integral

Often enough the numerical analyst is presented with the challenge of finding the definite integral of some function:

$$\int_a^b f(x)\mathrm{d}x.$$

In your golden years of Calculus, you learned the Fundamental Theorem of Calculus, which claims that if $f(x)$ is continuous, and $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

What you might not have been told in Calculus is there are some functions for which a closed form antiderivative does not exist or at least is not known to humankind. Nevertheless, you may find yourself in a situation where you have to evaluate an integral for just such an integrand. An approximation will have to do.

Upper and Lower Sums

We will review the definition of the Riemann integral of a function. A *partition* of an interval $[a, b]$ is some a finite ordered collection of nodes x_i:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Given such a partition, P, define the upper and lower bounds on each subinterval $[x_j, x_{j+1}]$ as follows:

$$\begin{aligned} m_i &= \inf \{ f(x) \mid x_i \le x \le x_{i+1} \} \\ M_i &= \sup \{ f(x) \mid x_i \le x \le x_{i+1} \} \end{aligned}$$

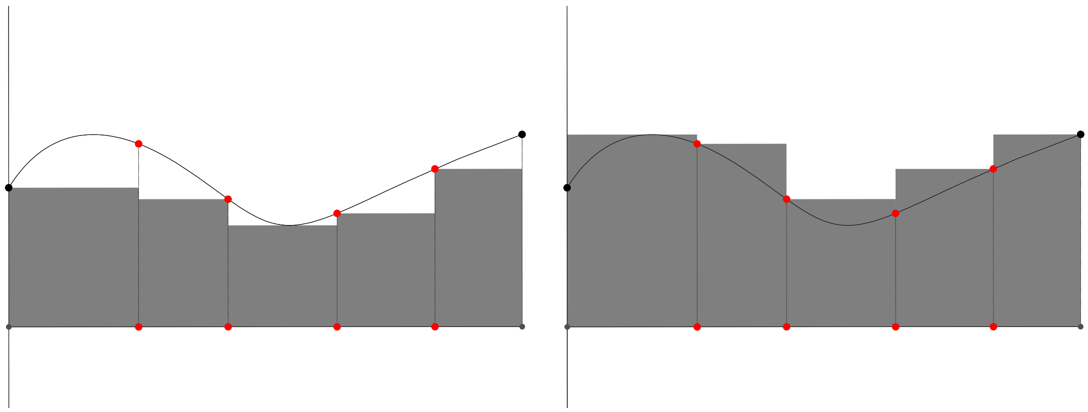
Then for this function f and partition P, define the upper and lower sums:

$$\begin{aligned} L(f, P) &= \sum_{i=0}^{n-1} m_i \left(x_{i+1} - x_i \right) \\ U(f, P) &= \sum_{i=0}^{n-1} M_i \left(x_{i+1} - x_i \right) \end{aligned}$$

We can interpret the upper and lower sums graphically as the sums of areas of rectangles defined by the function f and the partition P, as in Figure 1.

Notice a few things about the upper, lower sums:

(i) $L(f, P) \le U(f, P)$.



(a) The Lower Sum (b) The Upper Sum

Figure 1: The (a) lower, and (b) upper sums of a function on a given interval are shown. These approximations to the integral are the sums of areas of rectangles. Note that the lower sums are an underestimate, and the upper sums an overestimate of the integral.

(ii) If we switch to a "better" partition (*i.e.*, a finer one), we expect that $L(f, \cdot)$ increases and $U(f, \cdot)$ decreases.

The notion of integrability familiar from Calculus class (that is Riemann Integrability) is defined in terms of the upper and lower sums.

Definition 1. A function f is Riemann Integrable over interval $[a, b]$ if

$$\sup_P L(f, P) = \inf_P U(f, P),$$

where the supremum and infimum are over all partitions of the interval $[a, b]$. Moreover, in case $f(x)$ is integrable, we define the integral

$$\int_a^b f(x)\mathrm{d}x = \inf_P U(f, P),$$

You may recall the following

Theorem 2. Every continuous function on a closed bounded interval of the real line is Riemann Integrable (on that interval).

Continuity is sufficient, but not necessary. Consider the *Heaviside function*

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \leq x \end{cases}$$

This function is not continuous on any interval containing 0, but is Riemann Integrable on every closed bounded interval.

Consider also the *Dirichlet function*:

$$f(x) = \begin{cases} 0 & x \text{ rational} \\ 1 & x \text{ irrational} \end{cases}$$

For any partition P of any interval $[a, b]$, we have $L(f, P) = 0$, while $U(f, P) = 1$, so

$$\sup_P L(f, P) = 0 \neq 1 = \inf_P U(f, P),$$

so this function is *not* Riemann Integrable.

Approximating the Integral

Oddly enough, this definition gives an easy method of approximating an integral $\int_a^b f(x)\mathrm{d}x$. The method cuts the interval $[a, b]$ into a partition of n equal subintervals $x_i = a + \frac{b-a}{n}$, for $i = 0, 1, \ldots, n$. The algorithm then has to somehow find the supremum and infimum of $f(x)$ on each interval $[x_i, x_{i+1}]$. This is not feasible on a general black box function. However, if some information is known about the function, it becomes easier.

Consider for example, using this method on some function $f(x)$ which is *monotone increasing*. In this case, the infimum of $f(x)$ on each interval occurs at the leftmost endpoint, while the supremum occurs at the right hand endpoint. Thus for this partition, P, we have

$$\begin{aligned} L(f, P) &= \sum_{k=0}^{n-1} m_i \, |x_{k+1} - x_k| = \frac{|b-a|}{n} \sum_{k=0}^{n-1} f(x_k) \\ U(f, P) &= \sum_{k=0}^{n-1} M_i \, |x_{k+1} - x_k| = \frac{|b-a|}{n} \sum_{k=0}^{n-1} f(x_{k+1}) = \frac{|b-a|}{n} \sum_{k=1}^{n} f(x_k) \end{aligned}$$

We can easily estimate the error of this approximation. Because

$$L(f, P) \leq \int_a^b f(x)\mathrm{d}x \leq U(f, P),$$

we know that if we take the upper sum to be an approximation of our integral, then

$$\left| U(f, P) - \int_a^b f(x)\mathrm{d}x \right| \leq |U(f, P) - L(f, P)| = \frac{|b-a|}{n} \left[f(x_n) - f(x_0) \right] = \frac{|b-a| \left[f(b) - f(a) \right]}{n}$$