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§5.2 Trapezoid Rule

Recall the setting: we are trying to approximate the integral

$$\int_a^b f(x)\mathrm{d}x,$$

for some unpleasant or black box function $f(x)$; we *cannot* just find some antiderivative of $f(x)$ and use the Fundamental Theorem of Calculus.

A *Partition* of an interval $[a, b]$ is the ordered collection of $n + 1$ nodes

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b.$$

Most of the "rules" that we will use to approximate integrals use a partition of the interval; we need only describe how to approximate the integral over a single interval $[x_i, x_{i+1}]$.

Trapezoid Rule

The trapezoid rule approximates the integral

$$\int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x$$

by the (signed) area of the trapezoid through the points $(x_i, f(x_i)), (x_{i+1}, f(x_{i+1}))$, and with one side the segment from x_i to x_{i+1}. See Figure 1.

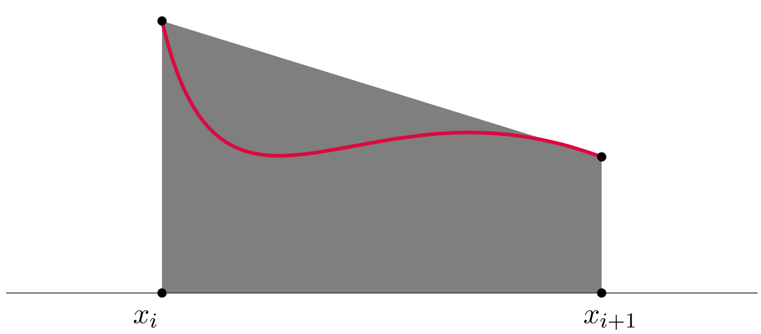

Figure 1: The trapezoid rule for approximating the integral is shown.

By old school math, we can find this signed area easily. This gives the trapezoid rule:

$$\int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x \approx (x_{i+1} - x_i)\,\frac{f(x_i) + f(x_{i+1})}{2}.$$

Then the integral over the entire interval is approximated by

$$\int_a^b f(x)\mathrm{d}x = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x \approx \frac{1}{2}\sum_{i=0}^{n-1} (x_{i+1} - x_i)\left[f(x_i) + f(x_{i+1})\right].$$

This is a *composite* rule; that is, the rule is determined by applying the same formula to each subinterval of the partition.

In practice the partition often consists of equally spaced points. Then $x_{i+1} - x_i = h = \frac{b-a}{n}$. The rule then becomes

$$\int_a^b f(x)\mathrm{d}x \approx \frac{h}{2}\sum_{i=0}^{n-1} \left[f(x_i) + f(x_{i+1})\right].$$

In your calculus class, you saw this in the less comprehensible form:

$$\int_a^b f(x)\mathrm{d}x \approx h\left[\frac{f(a) + f(b)}{2} + \sum_{i=1}^{n-1} f(x_i)\right].$$

Example 1. Approximate the integral

$$\int_0^2 \frac{1}{1+x^2}\mathrm{d}x$$

by a partition of equally spaced points, with $n = 2$.
We have $h = \frac{2-0}{2} = 1$, and $f(x_0) = 1, f(x_1) = \frac{1}{2}, f(x_2) = \frac{1}{5}$. Then the Trapezoid Rule gives the value

$$\frac{1}{2}\left[f(x_0) + f(x_1) + f(x_1) + f(x_2)\right] = \frac{1}{2}\left[1 + 1 + \frac{1}{5}\right] = \frac{11}{10}.$$

The actual value is $\arctan 2 \approx 1.107149$, thus we our approximation is correct to two decimal places.

How Good is the Trapezoid Rule?

We consider the Trapezoid rule for partitions of equal subintervals. Let $p_i(x)$ be the polynomial of degree ≤ 1 that interpolates $f(x)$ at x_i, x_{i+1}. Let

$$I_i = \int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x, \quad T_i = \int_{x_i}^{x_{i+1}} p_i(x)\mathrm{d}x = (x_{i+1} - x_i)\,\frac{p_i(x_i) + p_i(x_{i+1})}{2} = \frac{h}{2}\left(f(x_i) + f(x_{i+1})\right).$$

That's right: the trapezoidal rule approximates the integral of $f(x)$ over $[x_i, x_{i+1}]$ by the integral of $p_i(x)$ over the same interval.

Now recall our theorem on polynomial interpolation error. For $x \in [x_i, x_{i+1}]$, we have

$$f(x) - p_i(x) = \frac{1}{(2)!} f^{(2)}(\xi_x)(x - x_i)(x - x_{i+1}),$$

for some $\xi_x \in [x_i, x_{i+1}]$. Recall that ξ_x depends on x. To make things simpler, call it $\xi(x)$.

Now integrate:

$$I_i - T_i = \int_{x_i}^{x_{i+1}} f(x) - p_i(x)\mathrm{d}x = \frac{1}{2}\int_{x_i}^{x_{i+1}} f''(\xi(x))(x - x_i)(x - x_{i+1})\,\mathrm{d}x.$$

We will now attack the integral on the right hand side. Recall the following theorem:

Theorem 2 (Mean Value Theorem for Integrals). Suppose f is continuous, g is Riemann Integrable and does not change sign on $[\alpha, \beta]$. Then there is some $\zeta \in [\alpha, \beta]$ such that

$$\int_\alpha^\beta f(x)g(x)\mathrm{d}x = f(\zeta)\int_\alpha^\beta g(x)\mathrm{d}x.$$

We use this theorem on our integral. Note that $(x - x_i)(x - x_{i+1})$ is nonpositive on the interval of question, $[x_i, x_{i+1}]$. We assume continuity of $f''(x)$, and wave our hands to get continuity of $f''(\xi(x))$. Then we have

$$I_i - T_i = \frac{1}{2} f''(\xi)\int_{x_i}^{x_{i+1}} (x - x_i)(x - x_{i+1})\,\mathrm{d}x,$$

for some $\xi_i \in [x_i, x_{i+1}]$. By boring calculus and algebra, we find that

$$\int_{x_i}^{x_{i+1}} (x - x_i)(x - x_{i+1})\,\mathrm{d}x = -\frac{h^3}{6}.$$

This gives

$$I_i - T_i = -\frac{h^3}{12} f''(\xi_i),$$

for some $\xi_i \in [x_i, x_{i+1}]$.

We now sum over all subintervals to find the total error of the Trapezoid rule

$$E = \sum_{i=0}^{n-1} I_i - T_i = -\frac{h^3}{12}\sum_{i=0}^{n-1} f''(\xi_i) = -\frac{(b-a)\,h^2}{12}\left[\frac{1}{n}\sum_{i=0}^{n-1} f''(\xi_i)\right].$$

On the far right we have an average value, $\frac{1}{n}\sum_{i=0}^{n-1} f''(\xi_i)$, which lies between the least and greatest values of f'' on the inteval $[a, b]$, and thus by the IVT, there is some ξ which takes this value. So

$$E = -\frac{(b-a)\,h^2}{12} f''(\xi)$$

This gives us the theorem:

Theorem 3 (Error of the Trapezoid Rule). Let $f''(x)$ be continuous on $[a, b]$. Let T be the value of the trapezoid rule applied to $f(x)$ on this interval with a partition of uniform spacing, h, and let $I = \int_a^b f(x)\mathrm{d}x$. Then there is some $\xi \in [a, b]$ such that

$$I - T = -\frac{(b-a)\,h^2}{12} f''(\xi).$$

Note that this theorem tells us not only the magnitude of the error, but the sign as well. Thus if, for example, $f(x)$ is concave up and thus f'' is positive, then $I - T$ will be negative, *i.e.*, the Trapezoid Rule gives an *overestimate* of the integral I. See Figure 2.

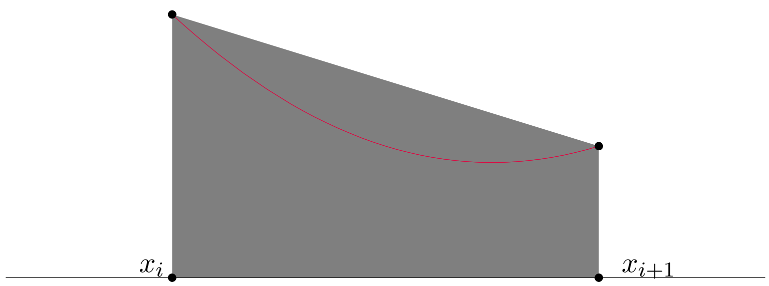

Figure 2: The trapezoid rule is an overestimate for a function which is concave up, *i.e.*, has positive second derivative.

Using the Error Bound

Consider the following examples:

Example 4. How many intervals are required to approximate the integral

$$\ln 2 = I = \int_0^1 \frac{1}{1+x}\mathrm{d}x$$

to within 1×10^{-10}?

We have $f(x) = \frac{1}{1+x}$, thus $f'(x) = -\frac{1}{(1+x)^2}$. And $f''(x) = \frac{2}{(1+x)^3}$. Thus $f''(\xi)$ is continuous and bounded by 2 on $[0, 1]$. If we use n equal subintervals then the theorem tells us our error will be

$$-\frac{1-0}{12}\left(\frac{1-0}{n}\right)^2 f''(\xi) = -\frac{f''(\xi)}{12n^2}.$$

To make this smaller than 1×10^{-10}, in absolute value, we need only take

$$\frac{1}{6n^2} \leq 1 \times 10^{-10},$$

and so $n \geq \sqrt{\frac{1}{6}} \times 10^5$ suffices. Because $f''(x)$ is positive on this interval, the Trapezoid Rule will be an overestimate.