Steven Pav SPAV@MATH.UCSD.EDU

§5.5 Gaussian Quadrature

The word *quadrature* refers to a method of approximating the integral of a function as the linear combination of the function at certain points, *i.e.*,

$$\int_a^b f(x)\mathrm{d}x \approx A_0 f(x_0) + A_1 f(x_1) + \ldots A_n f(x_n),$$

for some collection of nodes $\{x_i\}_{i=0}^n$, and weights $\{A_i\}_{i=0}^n$. Normally one finds the nodes and weights in a table somewhere; we expect a quadrature rule with more nodes to be more accurate in some sense–the tradeoff is in the number of evaluations of $f(\cdot)$. We will examine how these rules are created.

Determining Coefficients

Suppose that the nodes $\{x_i\}_{i=0}^n$ are given. An easy way to find "good" coefficients $\{A_i\}_{i=0}^n$ for these nodes is to rig them so the quadrature rule gives the integral of $p(x)$, the polynomial which interpolates $f(x)$ on these nodes. Recall

$$p(x) = \sum_{i=0}^n f(x_i)\ell_i(x),$$

where $\ell_i(x)$ is the i^{th} Lagrange polynomial. Thus our rigged approximation is the one that gives

$$\int_a^b f(x)\mathrm{d}x \approx \int_a^b p(x)\mathrm{d}x = \sum_{i=0}^n f(x_i)\int_a^b \ell_i(x)\mathrm{d}x.$$

If we let

$$A_i = \int_a^b \ell_i(x)\mathrm{d}x,$$

then we have a quadrature rule.

If $f(x)$ is a polynomial of degree $\leq n$ then $f(x) = p(x)$, and the quadrature rule is exact.

Example 1. Construct a quadrature rule on the interval $[0,4]$ using nodes $0,1,2$.
The nodes are given, we determine the coefficients by constructing the Lagrange Polynomials, and integrating them.

$$\begin{aligned}
\ell_0(x) &= \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{1}{2}(x-1)(x-2),\\
\ell_1(x) &= \frac{(x-0)(x-2)}{(1-0)(1-2)} = -(x)(x-2),\\
\ell_2(x) &= \frac{(x-0)(x-1)}{(2-0)(2-1)} = \frac{1}{2}(x)(x-1).
\end{aligned}$$

Then the coefficients are

$$\begin{aligned}
A_0 = \int_0^4 \ell_0(x)\mathrm{d}x &= \int_0^4 \frac{1}{2}(x-1)(x-2)\mathrm{d}x = \frac{8}{3}, \\
A_1 = \int_0^4 \ell_1(x)\mathrm{d}x &= \int_0^4 -(x)(x-2)\mathrm{d}x = -\frac{16}{3}, \\
A_2 = \int_0^4 \ell_2(x)\mathrm{d}x &= \int_0^4 \frac{1}{2}(x)(x-1)\mathrm{d}x = \frac{20}{3}.
\end{aligned}$$

Thus our quadrature rule is

$$\boxed{\int_0^4 f(x)\mathrm{d}x \approx \frac{8}{3}f(0) - \frac{16}{3}f(1) + \frac{20}{3}f(2).}$$

We expect this rule to be exact for a quadratic function $f(x)$. To illustrate this, let $f(x) = x^2 + 1$. By calculus we have

$$\int_0^4 x^2 + 1\,\mathrm{d}x = \frac{1}{3}x^3 + x\Big|_0^4 = \frac{64}{3} + 4 = \frac{76}{3}.$$

The approximation is

$$\int_0^4 x^2 + 1\,\mathrm{d}x \approx \frac{8}{3}\left[0+1\right] - \frac{16}{3}\left[1+1\right] + \frac{20}{3}\left[4+1\right] = \frac{76}{3}.$$

Gaussian Nodes

It would seem this is the best we can do: using $n+1$ nodes we can devise a quadrature rule that is exact for polynomials of degree $\leq n$ by calculating the coefficients from the Lagrange Polynomials. It turns out that by choosing the nodes in the right way, we can do far better. Gauss discovered that the right nodes to choose are the $n+1$ roots of the (nontrivial) polynomial, $q(x)$, of degree $n+1$ which has the property

$$\int_a^b x^k q(x)\mathrm{d}x = 0 \quad (0 \leq k \leq n).$$

(If you view the integral as an inner product, you could say that $q(x)$ is orthogonal to the polynomials x^k in the resultant inner product space, but that's just fancy talk.)

Suppose that we have such a $q(x)$–we will not prove existence or uniqueness. Let $f(x)$ be a polynomial of degree $\leq 2n+1$. We write

$$f(x) = p(x)q(x) + r(x).$$

Both $p(x), r(x)$ are of degree $\leq n$. Because of how we picked $q(x)$ we have

$$\int_a^b p(x)q(x)\mathrm{d}x = 0.$$

Thus

$$\int_a^b f(x)\mathrm{d}x = \int_a^b p(x)q(x)\mathrm{d}x + \int_a^b r(x)dx = \int_a^b r(x)dx.$$

Now suppose that the A_i are chosen by Lagrange Polynomials so the quadrature rule on the nodes x_i is exact for polynomials of degree $\leq n$. Then

$$\sum_{i=0}^{n} A_i f(x_i) = \sum_{i=0}^{n} A_i \left[p(x_i)q(x_i) + r(x_i)\right] \sum_{i=0}^{n} A_i r(x_i).$$

The last equality holds becuase the x_i are the roots of $q(x)$. Because of how the A_i are chosen we then have

$$\sum_{i=0}^{n} A_i f(x_i) = \sum_{i=0}^{n} A_i r(x_i) = \int_a^b r(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x.$$

Thus this rule is exact for $f(x)$. We have (or rather, Gauss has) made quadrature twice as good.

Theorem 2 (Gaussian Quadrature Theorem). Let x_i be the $n+1$ roots of a (nontrivial) polynomial, $q(x)$, of degree $n+1$ which has the property

$$\int_a^b x^k q(x)\mathrm{d}x = 0 \quad (0 \leq k \leq n)\,.$$

Let A_i be the coefficients for these nodes chosen by integrating the Lagrange Polynomials. Then the quadrature rule for this choice of nodes and coefficients is exact for polynomials of degree $\leq 2n+1$.

Determining Gaussian Nodes

We can determine the Gaussian nodes in the same way we determine coefficients. The example is illustrative

Example 3. Construct the two Gaussian nodes for a quadrature rule on the interval $[0, 2]$. The function $q(x)$ is of degree 2, and should be orthogonal to $1, x$. Let $q(x) = c_0+c_1x+c_2x^2$. Then we want

$$\int_0^2 1q(x)\mathrm{d}x = \int_0^2 xq(x)\mathrm{d}x = 0$$

So we want

$$\int_0^2 c_0 + c_1x + c_2x^2\mathrm{d}x = \int_0^2 c_0x + c_1x^2 + c_2x^3\mathrm{d}x = 0$$

We find that we have to solve the equations

$$\begin{aligned} 2c_0 + 2c_1 + \frac{8}{3}c_2 &= 0, \\ 2c_0 + \frac{8}{3}c_1 + 4c_2 &= 0. \end{aligned}$$

In later sections we will see how to solve equations of this kind. For now, we "guess" the answer $c_0 = 2, c_1 = -6, c_2 = 3$.

Then our nodes are the roots of $q(x) = 2 - 6x + 3x^2$. That is the roots

$$\frac{6 \pm \sqrt{36 - 24}}{6} = 1 \pm \frac{2\sqrt{3}}{3}.$$

These nodes are a bit ugly. Rather than construct the Lagrange Polynomials, we will use the *method of undetermined coefficients*. Remember, we want to construct A_0, A_1 such that

$$\int_0^2 f(x)\mathrm{d}x \approx A_0 f(1 - \frac{2\sqrt{3}}{3}) + A_1 f(1 + \frac{2\sqrt{3}}{3})$$

is exact for polynomial $f(x)$ of degree ≤ 1. It suffices to make this approximation exact for the "building blocks" of such polynomials, that is, for the functions 1 and x. That is, it suffices to find A_0, A_1 such that

$$\begin{aligned} \int_0^2 1\mathrm{d}x &= A_0 + A_1 \\ \int_0^2 x\mathrm{d}x &= A_0(1 - \frac{\sqrt{3}}{3}) + A_1(1 + \frac{\sqrt{3}}{3}) \end{aligned}$$

This gives the equations

$$\begin{aligned} 2 &= A_0 + A_1 \\ 2 &= A_0(1 - \frac{\sqrt{3}}{3}) + A_1(1 + \frac{\sqrt{3}}{3}) \end{aligned}$$

This is solved by $A_0 = A_1 = 1$.

Thus our quadrature rule is

$$\boxed{\int_0^2 f(x)\mathrm{d}x \approx f(1 - \frac{\sqrt{3}}{3}) + f(1 + \frac{\sqrt{3}}{3})}.$$

We expect this rule to be exact for cubic polynomials. You should try this with $f(x) = x^3$.

Reinventing the Wheel

While it is good to know the theory, it doesn't make sense in practice to recompute these things all the time. There are books full of quadrature rules; your textbook even lists a few. On first inspection, it would seem you would have to have a different rule for each interval $[a, b]$. This is not the case, as we shall see, and the rules in the book are normally for the interval $[-1, 1]$.

If you need to approximate the integral of $f(x)$ over $[a, b]$, and have a good rule for the interval $[-1, 1]$, consider the substitution:

$$x = \frac{b-a}{2}t + \frac{b+a}{2}, \quad \text{so} \quad \mathrm{d}x = \frac{b-a}{2}\mathrm{d}t.$$

Then

$$\int_a^b f(x)\mathrm{d}x = \int_{-1}^1 f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)\frac{b-a}{2}\mathrm{d}t.$$

Letting

$$g(t) = \frac{b-a}{2} f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right),$$

then if $f(x)$ is a polynomial, $g(t)$ is a polynomial of the same degree. We can then use the $[-1, 1]$ quadrature rule on $g(t)$.