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Administrative Shift

The textbook does an excellent job of describing things in greater detail than can be presented in class. I have come to the conclusion that there is little value in rewriting the textbook for these notes, and should instead concentrate on getting the main points down. Because my notes in the past have been so 'talky,' often I forget to write things on the board in class. What goes on the board in class should be the main points of the topic at hand–students never write down what you *say*, they write down what you write on the board. I want to spend more time presenting the right things in class, and on the board, and less time making these notes fully independent. Please let me know if this presents a major problem. For those that attend the class over the internet, please refer to the textbook when these notes make no sense. thanks.

§6.2 Smart Gaussian Elimination

Gaussian Elimination can fail when performed in the wrong order. If the algorithm selects a zero pivot, the multipliers are undefined, which is no good. We also saw that a pivot small in magnitude can cause failure. As here:

$$\begin{cases} \epsilon x_1 + x_2 &= 1 \\ x_1 + x_2 &= 2 \end{cases}$$

The Naive Algorithm solves this as

$$\begin{aligned} x_1 &= \frac{2 - \frac{1}{\epsilon}}{1 - \frac{1}{\epsilon}} = 1 - \frac{\epsilon}{1-\epsilon} \\ x_2 &= \frac{1 - x_2}{\epsilon} = \frac{1}{1-\epsilon} \end{aligned}$$

If ϵ is very small, then $\frac{1}{\epsilon}$ is enormous compared to both 1 and 2. With poor rounding, the algorithm solves x_2 as 1. Then it solves $x_1 = 0$.

The real solution is actually a bit different. Suppose the algorithm changed the order of the equations, then solved:

$$\begin{cases} x_1 + x_2 &= 2 \\ \epsilon x_1 + x_2 &= 1 \end{cases}$$

The Algorithm solves this as

$$\begin{aligned} x_2 &= \frac{1 - 2\epsilon}{1 - \epsilon} \\ x_1 &= 2 - x_2 \end{aligned}$$

There's no problem with rounding here.

The problem is not the small entry *per se*: Suppose we use an e.r.o. to scale the first equation, then use naïve G.E.:

$$\begin{cases} x_1 + \frac{1}{\epsilon}x_2 &= \frac{1}{\epsilon} \\ x_1 + x_2 &= 2 \end{cases}$$

This is still solved as

$$\begin{aligned} x_1 &= \frac{2 - \frac{1}{\epsilon}}{1 - \frac{1}{\epsilon}} \\ x_2 &= \frac{1 - x_2}{\epsilon}, \end{aligned}$$

and rounding is a problem.

Scaled Partial Pivoting

The naïve G.E. algorithm uses the rows 1, 2, ..., n-1 in order as pivot equations. As shown above, this can cause errors. Better is to pivot first on row ℓ_1, then row ℓ_2, etc, until finally pivoting on row ℓ_{n-1}, for some permutation $\{\ell_i\}_{i=1}^n$ of the integers 1, 2, ..., n. The strategy of *scaled partial pivoting* is to compute this permutation so that G.E. works well.

In light of our example, we want to pivot on an element which is not small compared to other elements in its row. So our algorithm first determines "smallness" by calculating a scale, row-wise:

$$s_i = \max_{1 \le j \le n} |a_{ij}| \,.$$

The scales are only computed once.

Then the first pivot, ℓ_1, is chosen to be the i such that

$$\frac{|a_{i,1}|}{s_i}$$

is maximized. The algorithm pivots on row ℓ_1, producing a bunch of zeros in the first column. Note that the algorithm should *not* rearrange the matrix–this takes too much work.

The second pivot, ℓ_2, is chosen to be the i such that

$$\frac{|a_{i,2}|}{s_i}$$

is maximized, but without choosing $\ell_2 = \ell_1$. The algorithm pivots on row ℓ_2, producing a bunch of zeros in the second column.

In the k^{th} step ℓ_k is chosen to be the i not among $\ell_1, \ell_2, \ldots, \ell_{k-1}$ such that

$$\frac{|a_{i,k}|}{s_i}$$

is maximized. The algorithm pivots on row ℓ_k, producing a bunch of zeros in the k^{th} column.

The slick way to implement this is to first set $\ell_i = i$ for $i = 1, 2, \ldots, n$. Then rearrange this vector in a kind of "bubble sort": when you find the index that should be ℓ_1, swap them, *i.e.*, find the j such that ℓ_j should be the first pivot and switch the values of ℓ_1, ℓ_j.

Then at the k^{th} step, search only those indices in the tail of this vector: *i.e.*, only among ℓ_j for $k \le j \le n$, and perform a swap.

An Example

We present an example of using scaled partial pivoting with G.E. It's hard to come up with an example where the numbers do not come out as ugly fractions. We'll look at a homework question. The book shows another example problem.

$$\left(\begin{array}{cccc|c} 2 & -1 & 3 & 7 & 15 \\ 4 & 4 & 0 & 7 & 11 \\ 2 & 1 & 1 & 3 & 7 \\ 6 & 5 & 4 & 17 & 31 \end{array}\right)$$

The scales are as follows: $s_1 = 7, s_2 = 7, s_3 = 3, s_4 = 17$.

We pick ℓ_1. It should be the index which maximizes $|a_{i1}|/s_i$. These values are:

$$\frac{2}{7}, \frac{4}{7}, \frac{2}{3}, \frac{6}{17}.$$

We pick $\ell_1 = 3$, and pivot:

$$\left(\begin{array}{cccc|c} 0 & -2 & 2 & 4 & 8 \\ 0 & 2 & -2 & 1 & -3 \\ 2 & 1 & 1 & 3 & 7 \\ 0 & 2 & 1 & 8 & 10 \end{array}\right)$$

We pick ℓ_2. It should *not* be 3, and should be the index which maximizes $|a_{i2}|/s_i$. These values are:

$$\frac{2}{7}, \frac{2}{7}, \frac{2}{17}.$$

We have a tie. In this case we pick the second row, *i.e.*, $\ell_2 = 2$. We pivot:

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 5 & 5 \\ 0 & 2 & -2 & 1 & -3 \\ 2 & 1 & 1 & 3 & 7 \\ 0 & 0 & 3 & 7 & 13 \end{array}\right)$$

The matrix is in permuted upper triangular form. We could proceed, but would get a zero multiplier, and no changes would occur.

If we did proceed we would have $\ell_3 = 4$. Then $\ell_4 = 1$. Our row permutation is $3, 2, 4, 1$. When we do back substitution, we work in this order *reversed* on the rows, solving x_4, then x_3, x_2, x_1.

We get $x_4 = 1$, so

$$\begin{aligned} x_3 &= \frac{1}{3}(13 - 7 * 1) = 2 \\ x_2 &= \frac{1}{2}(-3 - 1 * 1 + 2 * 2) = 0 \\ x_1 &= \frac{1}{2}(7 - 3 * 1 - 1 * 2 - 1 * 0) = 1 \end{aligned}$$

Another Example and A Real Algorithm

Sometimes we want to solve

$$\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$$

for a number of different vectors $\boldsymbol{b}$. It turns out we can run G.E. on the matrix A alone and come up with all the multipliers, which can then be used multiple times on different vectors $\boldsymbol{b}$. We illustrate with an example:

$$\mathsf{M}_0 = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 4 & 2 & 1 & 2 \\ 2 & 1 & 2 & 3 \\ 1 & 2 & 4 & 1 \end{pmatrix}, \quad \boldsymbol{\ell} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

The scale vector is $\boldsymbol{s} = \begin{bmatrix} 3 & 4 & 3 & 4 \end{bmatrix}^\top$.

Our scale choices are $\frac{1}{3}, \frac{4}{4}, \frac{2}{3}, \frac{1}{4}$. We choose $\ell_1 = 2$, and swap ℓ_1, ℓ_2. In the places where there would be zeros in the real matrix, we will put the multipliers. We will illustrate them here boxed:

$$\mathsf{M}_1 = \begin{pmatrix} \boxed{-\frac{1}{4}} & \frac{5}{2} & \frac{7}{4} & \frac{1}{2} \\ 4 & 2 & 1 & 2 \\ \boxed{-\frac{1}{2}} & 0 & \frac{3}{2} & 2 \\ \boxed{-\frac{1}{4}} & \frac{3}{2} & \frac{15}{4} & \frac{1}{2} \end{pmatrix}, \quad \boldsymbol{\ell} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}.$$

Our scale choices are $\frac{5}{6}, \frac{0}{3}, \frac{3}{8}$. We choose $\ell_2 = 1$, and no swap is needed:

$$\mathsf{M}_2 = \begin{pmatrix} \boxed{-\frac{1}{4}} & \frac{5}{2} & \frac{7}{4} & \frac{1}{2} \\ 4 & 2 & 1 & 2 \\ \boxed{-\frac{1}{2}} & \boxed{0} & \frac{3}{2} & 2 \\ \boxed{-\frac{1}{4}} & \boxed{-\frac{3}{5}} & \frac{27}{10} & \frac{1}{5} \end{pmatrix}, \quad \boldsymbol{\ell} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}.$$

Our scale choices are $\frac{1}{2}, \frac{27}{40}$. We choose $\ell_3 = 4$, and so swap ℓ_3, ℓ_4:

$$\mathsf{M}_3 = \begin{pmatrix} \boxed{-\dfrac{1}{4}} & \frac{5}{2} & \frac{7}{4} & \frac{1}{2} \\ 4 & 2 & 1 & 2 \\ \boxed{-\dfrac{1}{2}} & \boxed{0} & \boxed{-\dfrac{5}{9}} & \frac{17}{9} \\ \boxed{-\dfrac{1}{4}} & \boxed{-\dfrac{3}{5}} & \frac{27}{10} & \frac{1}{5} \end{pmatrix}, \quad \boldsymbol{\ell} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix}.$$

Now suppose we had to solve the linear system for $\boldsymbol{b} = \begin{bmatrix} 1 & 8 & 2 & -1 \end{bmatrix}^\top$.

We scale $\boldsymbol{b}$ by the multipliers in order: $\ell_1 = 2$, so, we sweep through the first column of M_3, picking off the boxed numbers (your computer doesn't really have boxed variables), and scaling $\boldsymbol{b}$ appropriately:

$$\begin{bmatrix} 1 \\ 8 \\ 2 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 1 \\ -2 \\ -3 \end{bmatrix}$$

This continues:

$$\begin{bmatrix} -1 \\ 1 \\ -2 \\ -3 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 1 \\ -2 \\ -\frac{12}{5} \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 1 \\ -\frac{2}{3} \\ -\frac{12}{5} \end{bmatrix}$$

We then perform a permuted backwards substitution:

$$\begin{aligned} x_4 &= \frac{-2}{3}\frac{9}{17} = \frac{-6}{17} \\ x_3 &= \frac{10}{27}\left(-\frac{12}{5} - \frac{1}{5}\frac{-6}{17}\right) = \ldots \\ x_2 &= \frac{2}{5}\left(-1 - \frac{1}{2}\frac{-6}{17} - \frac{7}{4}x_3\right) = \ldots \\ x_1 &= \frac{1}{4}\left(1 - 2\frac{-6}{17} - x_3 - 2x_2\right) = \ldots \end{aligned}$$

Fill in your own values here.