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§6.4 LU Factorization

We examined G.E. to solve the system

$$\mathsf{A}\boldsymbol{x} = \boldsymbol{b},$$

where A is a matrix:

$$\mathsf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}.$$

We want to show that G.E. actually factors A into lower and upper triangular parts, that is $\mathsf{A} = \mathsf{LU}$, where

$$\mathsf{L} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \ell_{21} & 1 & 0 & \cdots & 0 \\ \ell_{31} & \ell_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \cdots & 1 \end{bmatrix}, \qquad \mathsf{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}.$$

We call this a *LU Factorization* of A.

An Example

We consider solution of the following augmented form:

$$\left(\begin{array}{rrrr|r} 2 & 1 & 1 & 3 & 7 \\ 4 & 4 & 0 & 7 & 11 \\ 6 & 5 & 4 & 17 & 31 \\ 2 & -1 & 0 & 7 & 15 \end{array}\right) \tag{1}$$

The naïve G.E. reduces this to

$$\left(\begin{array}{rrrr|r} 2 & 1 & 1 & 3 & 7 \\ 0 & 2 & -2 & 1 & -3 \\ 0 & 0 & 3 & 7 & 13 \\ 0 & 0 & 0 & 12 & 18 \end{array}\right)$$

We are going to run the naïve G.E., and see how it is a LU Factorization. Since this is the naïve version, we first pivot on the first row. Our multipliers are $-2, -3, -1$. We pivot to get

$$\left(\begin{array}{cccc|c} 2 & 1 & 1 & 3 & 7 \\ 0 & 2 & -2 & 1 & -3 \\ 0 & 2 & 1 & 8 & 10 \\ 0 & -2 & -1 & 4 & 8 \end{array}\right)$$

Careful inspection shows that we've merely multiplied A and $\boldsymbol{b}$ by a lower triangular matrix M_1:

$$\mathsf{M}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

The first column entries are the e.r.o. multipliers for each row. Thus after the first pivot, it is like we are solving the system

$$\mathsf{M}_1\mathsf{A}\boldsymbol{x} = \mathsf{M}_1\boldsymbol{b}.$$

We pivot on the second row to get:

$$\left(\begin{array}{cccc|c} 2 & 1 & 1 & 3 & 7 \\ 0 & 2 & -2 & 1 & -3 \\ 0 & 0 & 3 & 7 & 13 \\ 0 & 0 & -3 & 5 & 5 \end{array}\right)$$

The multipliers are $-1, 1$. We can view this pivot as a multiplication by M_2, with

$$\mathsf{M}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

We are now solving

$$\mathsf{M}_2\mathsf{M}_1\mathsf{A}\boldsymbol{x} = \mathsf{M}_2\mathsf{M}_1\boldsymbol{b}.$$

We pivot on the third row, with a multiplier of 1. Thus we get

$$\left(\begin{array}{cccc|c} 2 & 1 & 1 & 3 & 7 \\ 0 & 2 & -2 & 1 & -3 \\ 0 & 0 & 3 & 7 & 13 \\ 0 & 0 & 0 & 12 & 18 \end{array}\right)$$

We have multiplied by M_3 :

$$\mathsf{M}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We are now solving

$$\mathsf{M}_3\mathsf{M}_2\mathsf{M}_1\mathsf{A}\boldsymbol{x} = \mathsf{M}_3\mathsf{M}_2\mathsf{M}_1\boldsymbol{b}.$$

But we have an upper triangular form, that is, if we let

$$\mathsf{U} = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

Then we have

$$\begin{aligned} \mathsf{M}_3\mathsf{M}_2\mathsf{M}_1\mathsf{A} &= \mathsf{U}, \\ \mathsf{A} &= (\mathsf{M}_3\mathsf{M}_2\mathsf{M}_1)^{-1}\,\mathsf{U}, \\ \mathsf{A} &= \mathsf{M}_1{}^{-1}\mathsf{M}_2{}^{-1}\mathsf{M}_3{}^{-1}\mathsf{U}, \\ \mathsf{A} &= \mathsf{L}\mathsf{U}. \end{aligned}$$

We are hoping that L is indeed lower triangular, and has ones on the diagonal. It turns out that the inverse of each M_i matrix has a nice form. We write them here:

$$\begin{aligned} \mathsf{L} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{aligned}$$

This is really crazy: the matrix L looks to be composed of ones on the diagonal and multipliers under the diagonal.

Now we check to see if we made any mistakes:

$$\begin{aligned} \mathsf{L}\mathsf{U} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 3 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 12 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & 1 & 3 \\ 4 & 4 & 0 & 7 \\ 6 & 5 & 4 & 17 \\ 2 & -1 & 0 & 7 \end{bmatrix} = \mathsf{A}. \end{aligned}$$

Using LU Factorizations

We see that the G.E. algorithm can be used to actually calculate the LU factorization. We will look at this in more detail in another example. We now examine how we can use the LU factorization to solve the equation

$$\mathsf{A}\boldsymbol{x} = \boldsymbol{b},$$

Since we have $\mathsf{A} = \mathsf{LU}$, we first solve

$$\mathsf{L}\boldsymbol{z} = \boldsymbol{b},$$

then solve

$$\mathsf{U}\boldsymbol{x} = \boldsymbol{z}.$$

Since L is lower triangular, we can solve for $\boldsymbol{z}$ with a *forward* substitution. Similarly, since U is upper triangular, we can solve for $\boldsymbol{x}$ with a back susbsitution. We drag out the previous example (which we never got around to solving):

$$\left(\begin{array}{rrrr|r} 2 & 1 & 1 & 3 & 7 \\ 4 & 4 & 0 & 7 & 11 \\ 6 & 5 & 4 & 17 & 31 \\ 2 & -1 & 0 & 7 & 15 \end{array}\right)$$

We had found the LU factorization of A as

$$\mathsf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 3 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

So we solve

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix} \boldsymbol{z} = \begin{bmatrix} 7 \\ 11 \\ 31 \\ 15 \end{bmatrix}$$

We get

$$\boldsymbol{z} = \begin{bmatrix} 7 \\ -3 \\ 13 \\ 18 \end{bmatrix}$$

Now we solve

$$\begin{bmatrix} 2 & 1 & 1 & 3 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 12 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 7 \\ -3 \\ 13 \\ 18 \end{bmatrix}$$

We get the ugly solution

$$\boldsymbol{z} = \begin{bmatrix} \frac{37}{24} \\ \frac{-17}{12} \\ \frac{5}{6} \\ \frac{3}{2} \end{bmatrix}$$

Some Theory

We aren't doing much proving here. The following theorem has an ugly proof in the textbook:

Theorem 1. If A is an $n \times n$ matrix, and naïve Gaussian Elimination does not encounter a zero pivot, then the algorithm generates a LU factorization of A, where L is the lower triangular part of the output matrix, and U is the upper triangular part.

This theorem relies on us using the fancy version of G.E., which saves the multipliers in the spots where there should be zeros. In our discussion of it in class, I put the negative multipliers in those spots; these are the boxed numbers in the matrix example from the previous lecture. I guess it pays to look ahead. If correctly implemented, then, L is the lower triangular part but with ones put on the diagonal.

This theorem is proven in the book. This appears to me to be a case of something which can be better illustrated with an example or two and some informal investigation. The proof is an unillustrating index-chase–read it at your own risk.

Computing Inverses

We consider finding the inverse of A. Since

$$\mathsf{A}\mathsf{A}^{-1} = \mathsf{I},$$

then the j^{th} column of the inverse A^{-1} solves the equation

$$\mathsf{A}\boldsymbol{x} = \boldsymbol{e_j},$$

where $\boldsymbol{e_j}$ is the column matrix of all zeros, but with a one in the j^{th} position.

Thus we can find the inverse of A by running n linear solves. Obviously we are only going to run G.E. once, to put the matrix in LU form, then run n solves using forward and backward substitutions.