2003/11/21 **M174 : Numerical Methods in Science and Engineering**
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§8.1 Ordinary Differential Equations

We want to solve ordinary differential equations, that is find some $x(t)$ such that

$$\begin{cases} \frac{dx(t)}{dt} = f\left(t, x(t)\right), \\ x(a) = c. \end{cases}$$

When $f\left(t, x(t)\right)$ is some function that is independent of its second variable, for example

$$f(t, x) = t^2 - t,$$

the problem becomes an exercise in integral calculus. In this case we have to solve

$$\begin{cases} \frac{dx(t)}{dt} = t^2 - t, \\ x(a) = c. \end{cases}$$

The solution is $x(t) = \frac{1}{3}t^3 - \frac{1}{2}t^2 + K$, where K is chosen such that $x(a) = C$.

The problem can be considerably more difficult when f depends on its second variable.

Integration and 'Stepping'

We attempt to solve the ODE by integrating both sides. That is

$$\begin{aligned} \frac{dx(t)}{dt} &= f\left(t, x(t)\right), \quad \text{yields} \\ \int_t^{t+h} \mathrm{d}x &= \int_t^{t+h} f\left(r, x(r)\right) \mathrm{d}r, \quad \text{thus} \\ x(t+h) &= x(t) + \int_t^{t+h} f\left(r, x(r)\right) \mathrm{d}r. \end{aligned}$$

If we can approximate the integral then we have a way of 'stepping' from t to $t + h$, *i.e.*, if we have a good approximate of $x(t)$ we can approximate $x(t + h)$.

If we use the left-hand rectangle rule for approximating integrals, we get

$$x(t+h) \approx x(t) + hf(t, x(t)).$$

This is Euler's Method.

Trapezoid rule gives

$$x(t+h) \approx x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, x(t+h))\right].$$

But we cannot evaluate this exactly, since $x(t + h)$ appears on both sides of the equation. And it is embedded on the right hand side. Bummer. But if we could approximate it, say by using Euler's Method, maybe the formula would work. This is the idea behind the Runge-Kutta method (we cover this in §8.2).

Taylor's Series Methods

We see that our more accurate integral approximations will be useless since they require information we do not know, *i.e.*, evaluations of $f(t,x)$ for yet unknown x values. Thus we fall back on Taylor's Theorem. We can also view this as using integral approximations where all information comes from the left-hand endpoint. kinda.

We recall that for analytic x

$$x(t+h) = x(t) + hx'(t) + \frac{1}{2}h^2x''(t) + \ldots + \frac{1}{m!}h^m x^{(m)}(t) + \ldots$$

If we use the approximation

$$x(t+h) \approx x(t) + hx'(t) + \frac{1}{2}h^2x''(t) + \ldots + \frac{1}{m!}h^m x^{(m)}(t).$$

to solve the ODE, this is called a *Taylor's series method of order* m.

Again we will use the idea of 'stepping.' h is the step width.

Euler's Method

When $m = 1$ we recover Euler's Method:

$$x(t+h) = x(t) + hx'(t) = x(t) + hf(t, x(t)).$$

We look at this graphically in class for the ODE

$$\begin{cases} \frac{dx(t)}{dt} = x, \\ x(0) = 1. \end{cases}$$

The actual solution is $x(t) = e^t$. Euler's method will underestimate $x(t)$ because the curvature of the actual $x(t)$ is positive, and thus the function is always above its linearization. We see graphically that eventually the Euler's method approximation is very poor.

Higher Order Methods

Consider the problem

$$\begin{cases} \frac{dx(t)}{dt} = x + e^x, \\ x(0) = 1. \end{cases}$$

We use a Taylor's Series method with $m = 3$.

$$\begin{aligned} x'(t) &= 1 + e^x \\ x''(t) &= e^x \\ x'''(t) &= e^x \end{aligned}$$

We then have the step:

$$x(t+h) \approx x(t) + hx'(t) + \frac{1}{2}h^2x''(t) + \frac{1}{3}h^3x'''(t) = x(t) + h + \left(h + \frac{1}{2}h^2 + \frac{1}{3}h^3\right)e^{x(t)}.$$

Errors

There are two types of errors: truncation and roundoff. Truncation error is the error of truncating the Taylor's series after the m^{th} term. You can see that the truncation error is $\mathcal{O}\left(h^{m+1}\right)$. Roundoff error occurs because of the limited precision of computer arithmetic.

Both of these kinds of error can accumulate: Since we step from $x(t)$ to $x(t+h)$, an error in the value of $x(t)$ can cause a greater error in the value of $x(t+h)$. It turns out that for some ODEs and some methods, this does not happen. This brings us to the idea of stability