

§8.2 Runge Kutta Methods

Recall the ODE problem: find some $x(t)$ such that

$$\begin{cases} \frac{dx(t)}{dt} = f(t, x(t)), \\ x(a) = c, \end{cases}$$

where f, a, c are given.

Recall that the Taylor's series method has problems: either you are using the first order method (Euler's method), which suffers inaccuracy, or you are using a higher order method which requires evaluations of higher order derivatives of $x(t)$, *i.e.*, $x''(t), x'''(t)$, etc. This makes these methods less useful for the general setting of f being a blackbox function. The Runge Kutta methods (don't ask me how it's pronounced) seek to resolve this.

Taylor's Series Redux

Like all good numerical analysis, we fall back on Taylor's series. In this case we rely on the 2-dimensional version:

$$f(x+h, y+k) = \sum_{i=0}^{\infty} \frac{1}{i!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f(x, y).$$

The thing in the middle is an operator on $f(x, y)$. The partial derivative operators are interpreted exactly as if they were algebraic terms, that is:

$$\begin{aligned} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^0 f(x, y) &= f(x, y), \\ \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^1 f(x, y) &= h \frac{\partial f(x, y)}{\partial x} + k \frac{\partial f(x, y)}{\partial y}, \\ \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 f(x, y) &= h^2 \frac{\partial^2 f(x, y)}{\partial x^2} + 2hk \frac{\partial f(x, y)}{\partial x \partial y} + k^2 \frac{\partial^2 f(x, y)}{\partial y^2}, \\ &\vdots \end{aligned}$$

There is a truncated version of Taylor's series:

$$f(x+h, y+k) = \sum_{i=0}^n \frac{1}{i!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f(x, y) + \frac{1}{(n+1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f(\bar{x}, \bar{y}).$$

where (\bar{x}, \bar{y}) is a point on the line segment with endpoints (x, y) and $(x+h, y+k)$.

For practice, we will write this for $n = 1$:

$$f(x+h, y+k) = f(x, y) + hf_x(x, y) + kf_y(x, y) + \frac{1}{2} (h^2 f_{xx}(\bar{x}, \bar{y}) + 2hk f_{xy}(\bar{x}, \bar{y}) + k^2 f_{yy}(\bar{x}, \bar{y}))$$

Runge Kutta Methods

We now introduce the R-K method for stepping from $x(t)$ to $x(t+h)$. We suppose that α, β, w_1, w_2 are fixed constants. We then compute:

$$\begin{aligned}K_1 &= hf(t, x) \\K_2 &= hf(t + \alpha h, x + \beta K_1).\end{aligned}$$

Then we approximate:

$$x(t+h) \leftarrow x(t) + w_1 K_1 + w_2 K_2.$$

We now examine the “proper” choices of the constants. Note that $w_1 = 1, w_2 = 0$ corresponds to Euler’s method, and does not require computation of K_2 . We will pick another choice.

Also notice that the definition of K_2 should be related to Taylor’s theorem in two dimensions. Let’s look at it:

$$\begin{aligned}K_2/h &= f(t + \alpha h, x + \beta K_1) \\&= f(t + \alpha h, x + \beta hf(t, x)) \\&= f + \alpha h f_t + \beta h f f_x + \frac{1}{2} (\alpha^2 h^2 f_{tt}(\bar{t}, \bar{x}) + \alpha \beta h^2 f_{tx}(\bar{t}, \bar{x}) + \beta^2 h^2 f^2 f_{xx}(\bar{t}, \bar{x})).\end{aligned}$$

Now reconsider our step:

$$\begin{aligned}x(t+h) &= x(t) + w_1 K_1 + w_2 K_2 \\&= x(t) + w_1 hf + w_2 hf + w_2 \alpha h^2 f_t + w_2 \beta h^2 f f_x + \mathcal{O}(h^3). \\&= x(t) + (w_1 + w_2) hx'(t) + w_2 h^2 (\alpha f_t + \beta f f_x) + \mathcal{O}(h^3). \\&= x(t) + (w_1 + w_2) hx'(t) + w_2 h^2 (\alpha f_t + \beta f f_x) + \mathcal{O}(h^3).\end{aligned}$$

If we happen to choose our constants such that

$$w_1 + w_2 = 1, \quad \alpha w_2 = \frac{1}{2} = \beta w_2,$$

then we get

$$\begin{aligned}x(t+h) &= x(t) + hx'(t) + \frac{1}{2} h^2 (f_t + f f_x) + \mathcal{O}(h^3) \\&= x(t) + hx'(t) + \frac{1}{2} h^2 x''(t) + \mathcal{O}(h^3),\end{aligned}$$

i.e., our choice of the constants makes the approximate $x(t+h)$ good up to a $\mathcal{O}(h^3)$ term, because we end up with the Taylor’s series expansion up to that term. cool.

The usual choice of constants is $\alpha = \beta = 1, w_1 = w_2 = \frac{1}{2}$. This gives the *second order Runge Kutta method*:

$$x(t+h) \leftarrow x(t) + \frac{h}{2} f(t, x) + \frac{h}{2} f(t+h, x + hf(t, x)).$$

Another choice is $\alpha = \beta = 2/3, w_1 = 1/4, w_2 = 3/4$. This gives

$$x(t+h) \leftarrow x(t) + \frac{h}{4}f(t,x) + \frac{3h}{4}f\left(t + \frac{2h}{3}, x + \frac{2h}{3}f(t,x)\right).$$

The R-K method of order two has error term $\mathcal{O}(h^3)$. Sometimes this is not enough and higher-order R-K methods are used.

You can look in your textbook for the R-K method of order 4, which has error $\mathcal{O}(h^5)$. It requires four evaluations of f at funny places.

The R-K method can be extrapolated to higher orders. However, the number of function evaluations grows faster than the accuracy of the method. Thus the methods of order higher than four are normally not used.

We already saw that $w_1 = 1, w_2 = 0$ corresponds to Euler's Method. We consider now the case $w_1 = 0, w_2 = 1$. The method becomes

$$x(t+h) \leftarrow x(t) + hf\left(t + \frac{h}{2}, x + \frac{h}{2}f(t,x)\right).$$

This is called the *modified Euler's Method*. Note this gives a *different* value than if Euler's Method was applied twice with step size $h/2$.

Examples

Example 1. Consider the ODE:

$$\begin{cases} x' = (tx)^3 - \left(\frac{x}{t}\right)^2 \\ x(1) = 1 \end{cases}$$

Use $h = 0.1$ to compute $x(1.1)$ using both Taylor's Series Methods and R-K methods of order 2.