Homework 1

September 26, 2003

Homework 1 consists of the following questions from the text:

- §1.2 # 1, 8, 11, 13, 28, 44.
- §2.3 # 1, 3, 10, 11, 12.
- §3.1 # 5, 6, 8, 13.

Through special arrangement with the textbook's publisher, we list those questions here:

§1.2 # 1 The Taylor series for $(1+x)^n$ is known as the **binomial series.** It states that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \quad (x^2 < 1)$$

Derive the series and, for $n = 2, n = 3$, and $n = \frac{1}{2}$ give its particular form. Then use the last form to compute $\sqrt{1.0001}$ correct to 15 decimal places (rounded).

§1.2 # 8 Determine how many terms are needed to compute e correctly to 15 decimal places (rounded) using the series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad (|x| < \infty)\,.$$

§1.2 # 11 What is the Taylor series for $\ln(1-x)$? What is the series for $\ln\left[(1+x)/(1-x)\right]$.

§1.2 # 13 Use the Alternating Series Theorem to determine the number of terms in the series

$$\ln 1 + x = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{k=1} \infty \, (-1)^k \, \frac{x^k}{k} \quad (-1 < x \leq 1)\,.$$

needed to compute $\ln 1.1$ with error less than $\frac{1}{2} \times 10^{-8}$.

§1.2 # 28 By using Taylor's theorem, one can be sure that for all x that satisfy $|x| < \frac{1}{2}$, $\left|\cos x - (1 - x^2/2)\right|$ is less than or equal to what numerical value?

§1.2 # 44 How many terms are needed in the series

$$\operatorname{arccot} x = \frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \dots$$

to compute $\operatorname{arccot} x$ for $x^2 < 1$ accurate to 12 decimal places (rounded)?

§2.3 # 1 How can values of $f(x) = \sqrt{x+4} - 2$ be computed accurately when x is small?

§2.3 # 3 What is a good way to compute values of the function $f(x) = e^x - e$ if full machine precision is needed? *Note:* there is a difficulty when $x = 1$.

§2.3 # 10 For some values of x, the function $f(x) = \sqrt{x^2+1} - x$ cannot be accurately computed by using this formula. Explain and find a way around this difficulty.

§2.3 # 11 The inverse hyperbolic sine is given by $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$ Show how to avoid loss of significance in computing $f(x)$ when x is negative. *Hint:* Find and exploit the relationship between $f(x)$ and $f(-x)$.

§2.3 # 12 On most computers a highly accurate routine for $\cos x$ is provided. It is proposed to base a routine for $\sin x$ on the formula $\sin x = \pm\sqrt{1 - \cos^2 x}$. From the standpoint of precision (not efficiency), what problems do you foresee and how can they be avoided if we insist on using the routine for $\cos x$.

§3.1 # 5 Give an example of a function for which the bisection method does *not* converge linearly.

§3.1 # 6 Draw a graph of a function that is discontinuous yet that the bisection method converges. Repeat, getting a function for which it diverges.

§3.1 # 8 If $a = 0.1$, and $b = 1.0$ how many steps of the bisection method are needed to determine the root of a function with an error of at most $\frac{1}{2} \times 10^{-8}$?

§3.1 # 13 Try to devise a stopping criterion for the bisection method to guarantee that the root is determined with *relative* error at most ϵ.