2004/10/13	**M174 : Numerical Methods in Science and Engineering**
Steven Pav	SPAV@MATH.UCSD.EDU
Exam 1	**§1.1–1.3, 3.1–3.4, 4.1–4.3**

Instructions: Read all instructions carefully. Write your name and student number on your answer sheets. Clearly indicate your answers & show all your work. For many problems partial credit is available.

[Definitions] Answer *no more than two* of the following.

P1 (**12 pnts**) Clearly state Taylor's Theorem for $f(x+h)$. Include all hypotheses, and state the conditions on the variable that appears in the error term.

P2 (**12 pnts**) Write the general form of the iterated solution to the problem $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$. Let Q be your "splitting matrix," and use the factor ω. Your solution should look something like:

$$??\boldsymbol{x}^{(k)} = ??\boldsymbol{x}^{(k-1)} + ?? \qquad \text{or} \qquad \boldsymbol{x}^{(k)} = ??\boldsymbol{x}^{(k-1)} + ??$$

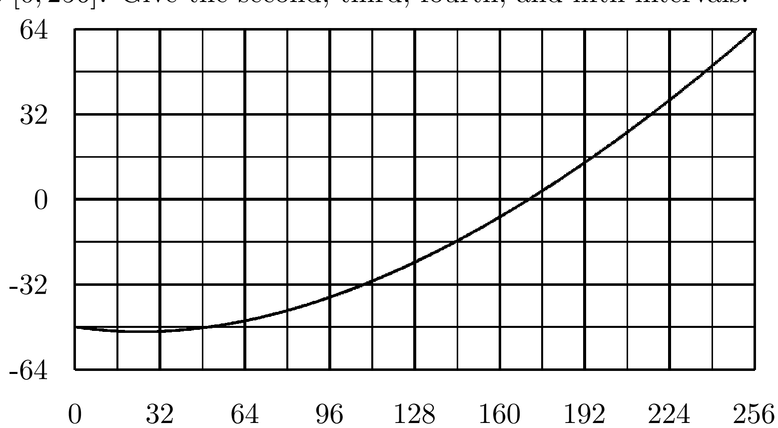
For one of the following variants, describe what Q is, in relation to A : Richardson's, Jacobi, Gauss-Seidel.

P3 (**12 pnts**) Write the iteration for Newton's Method *or* the Secant Method for finding a root to the equation $f(x) = 0$. Your solution should look something like:

$$x_{k+1} = ?? + \frac{??}{??}$$

[Problems] Answer *no more than four* of the following.

P4 (**14 pnts**) Perform bisection on the function graphed here. Let the first interval be $[0, 256]$. Give the second, third, fourth, and fifth intervals.



P5 (**14 pnts**) Give a $\mathcal{O}\left(h^4\right)$ approximation to $\sin\left(h + \pi/2\right)$. Find a reasonable C such that the truncation error is no greater than Ch^4.

P6 (**14 pnts**) Use Newton's Method to find a good approximation to $\sqrt[3]{24}$. Use $x_0 = 3$. That is, iteratively define a sequence with $x_0 = 3$, such that $x_k \to \sqrt[3]{24}$.

Exam continues on reverse of page. ↻

P7 (**14 pnts**) One of the roots of $x^2 - bx + c = 0$ as found by the quadratic formula is subject to subtractive cancellation when $|b| \gg |c|$. Which root is it? Rewrite the expression for that root to eliminate subtractive cancellation.

P8 (**14 pnts**) Find the LU decomposition of the following matrix by Gaussian Elimination with naïve pivoting.

$$\begin{bmatrix} 3 & 2 & 4 \\ -6 & 1 & -6 \\ 12 & -12 & 10 \end{bmatrix}$$

P9 (**14 pnts**) Consider the linear equation:

$$\begin{bmatrix} 6 & 1 & 1 \\ 2 & 4 & 0 \\ 1 & 2 & 6 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 5 \\ -6 \\ 3 \end{bmatrix}$$

Letting $\boldsymbol{x}^{(0)} = [1\ 1\ 1]^\top$, find $\boldsymbol{x}^{(1)}$ using either the Gauss-Seidel or Jacobi iterative schemes. Use weighting $\omega = 1$. You must indicate which scheme you are using. Show all your work.

[Questions] Answer *only one* of the following.

P10 (**20 pnts**) Suppose that the eigenvalues of A are in $[\alpha, \beta]$ with $0 < \alpha < \beta$. Find the ω that minimizes $\left\|\mathsf{I} - \omega \mathsf{A}^2\right\|_2$. What is the minimal value of $\left\|\mathsf{I} - \omega \mathsf{A}^2\right\|_2$ for this optimal ω? For full credit you need to make some argument *why* this ω minimizes this norm.

P11 (**20 pnts**) Let $f(x) = 0$ have a unique root r. Recall that the form of the error for Newton's Method is

$$e_{k+1} = \frac{-f''(\xi_k)}{2f'(x_k)} e_k^2,$$

where $e_k = r - x_k$. Suppose that $f(x)$ has the properties that $f'(x) \geq m > 0$ for all x, and $|f''(x)| \leq M$ for all x. Find $\delta > 0$ such that $|e_k| < \delta$ insures that $x_n \to r$. Justify your answer.

P12 (**20 pnts**) Suppose that $f(r) = 0 = f'(r) \neq f''(r)$, and $f''(x)$ is continuous. Letting $e_k = x_k - r$, show that for Newton's Method,

$$e_{k+1} \approx \frac{1}{2} e_k,$$

when $|e_k|$ is sufficiently small. (*Hint:* Use Taylor's Theorem twice.) Is this faster or slower convergence than for Newton's Method with a simple root?