Exam 1 Preparation

The first midterm exam is Monday October 25, during the class period. *You must bring a blue book to the exam.* The exam covers §1.1–1.3,3.1–3.4,4.1–4.3. You should be prepared to answer at least the following questions:

1. (§1.1) State Taylor's Theorem for $f(x+h)$. Give all the necessary conditions on f.

2. (§1.1) Give a $\mathcal{O}\left(h^5\right)$ approximation to $\sin(h)$ via Taylor's Theorem.

3. (§1.1) Give a $\mathcal{O}\left(h^3\right)$ approximation to $\ln(1+h)$ via Taylor's Theorem.

4. (§1.1) Calculate $\cos(\pi/2 + 0.001)$ to within 8 decimal places by using the Taylor's expansion.

5. (§1.2) Rewrite the expression $\sqrt{x+1} - 1$ to eliminate subtractive cancellation when $x \approx 0$.

6. (§1.2) Rewrite the quadratic formula for solving $ax^2 + bx + c = 0$ to eliminate potential subtractive cancellation of one of the roots.

7. (§1.3) Define what it means for $\lambda, \boldsymbol{x}$ to be eigenvalue and eigenvector of matrix A.

8. (§1.3) Suppose that one of the eigenvalues of A is zero. What does this say about A?

9. (§1.3) Suppose that the eigenvalues of A are $1, 10, 100$. Give the eigenvalues of $\mathsf{B} = 3\mathsf{A}^3 - 4\mathsf{A}^2 + \mathsf{I}$. Show that B is singular.

10. (§1.3) Find the eigenvalues and eigenvectors of the diagonal matrix:

$$\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

11. (§1.3) Define the norm of a vector, $\|\boldsymbol{x}\|_2$.

12. (§1.3) Define the matrix norm, $\|\mathsf{A}\|_2$.

13. (§1.3) Using the definition of matrix norm, prove that $\|\mathsf{A}\boldsymbol{x}\|_2 \le \|\mathsf{A}\|_2 \|\boldsymbol{x}\|_2$. (*Hint:* you should consider separately the cases when $\boldsymbol{x} = \boldsymbol{0}$, and when it does not.)

14. (§1.3) What is the norm of $\boldsymbol{x} = [3\ 5\ 10]^\top$?

15. (§1.3) What is the norm of

$$\mathsf{A} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1/2 & 0 & \cdots & 0 \\ 0 & 0 & 1/3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1/n \end{bmatrix}?$$

16. (§1.3) If $\|\boldsymbol{x}\|_2 = 0$, what does this say about vector $\boldsymbol{x}$?

17. (§1.3) If $\|\mathsf{A}\|_2 = 0$, what does this say about matrix A?

18. (§3.1) About how many flops are required to find the LU decomposition of an $n \times n$ matrix by Gaussian Elimination?

19. (§3.1) Perform back substitution to solve the equation

$$\begin{bmatrix} 1 & 3 & 5 & 3 \\ 0 & 2 & 4 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

20. (§3.1) About how many flops are required to perform a back substitution?

21. (§3.2) Under the strategy of scaled partial pivoting, which row of the following matrix will be the first pivot row?

$$\begin{bmatrix} 10 & 17 & -10 & 0.1 & 0.9 \\ -3 & 3 & -3 & 0.3 & -4 \\ 0.3 & 0.1 & 0.01 & -1 & 0.5 \\ 2 & 3 & 4 & -3 & 5 \\ 10 & 100 & 1 & 0.1 & 0 \end{bmatrix}$$

22. (§3.3) Find the LU decomposition of the following matrix by Gaussian Elimination with naïve pivoting:

$$\begin{bmatrix} -6 & 6 & 5 \\ 3 & 1 & -2 \\ 12 & 1 & 2 \end{bmatrix}$$

23. (§3.4) Write the general form of the iterated solution to the problem $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$. Let Q be your "splitting matrix," and use the factor ω. Your solution should look something like:

$$??\boldsymbol{x}^{(k)} = ??\boldsymbol{x}^{(k-1)} + ?? \qquad \text{or} \qquad \boldsymbol{x}^{(k)} = ??\boldsymbol{x}^{(k-1)} + ??$$

For one of the following variants, describe what Q is, in relation to A : Richardson's, Jacobi, Gauss-Seidel.

24. (§3.4) Derive the error formula

$$\boldsymbol{e}^{(k)} = \left(\mathsf{I} - \omega\mathsf{Q}^{-1}\mathsf{A}\right)\boldsymbol{e}^{(k-1)}$$

25. (§3.4) How does $\left\|\mathsf{I} - \omega\mathsf{Q}^{-1}\mathsf{A}\right\|_2$ affect convergence of the iterated solutions method? How is this related to the eigenvalues of $\mathsf{I} - \omega\mathsf{Q}^{-1}\mathsf{A}$? What is the purpose of ω in the iterative methods?

26. (§3.4) Suppose the eigenvalues of A are in $[\alpha, \beta]$ with $0 < \alpha < \beta$. Which ω should one pick to get the best convergence performance of Richardson's Iterations for this matrix? For which matrix do you expect faster convergence: A_1 with eigenvalues in $[10, 20]$ or A_2 with eigenvalues in $[1010, 1020]$? Why?

27. (§3.4) Consider the equation

$$\begin{bmatrix} 1 & 3 & 5 \\ -2 & 2 & 4 \\ 4 & -3 & -4 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} -5 \\ -6 \\ 10 \end{bmatrix}$$

Letting $\boldsymbol{x}^{(0)} = [1\ 1\ 0]^\top$, find the iterate $\boldsymbol{x}^{(1)}$ by one step of Richardson's Method. Or Jacobi. Or Gauss Seidel.

28. (§3.4) Compare and contrast the three iterative solvers: Richardson's, Jacobi, Gauss-Seidel. Discuss efficiency, convergence, implementation, etc.

29. (§4.1) Describe the bisection method for finding the root to $f(x) = 0$, on a given interval $[a, b]$.

30. (§4.1) Consider bisection for finding the root to $\cos x = 0$. Let the initial interval I_0 be $[0, 2]$. What is the next interval considered, call it I_1? What is I_2? I_6?

31. (§4.2) Describe Newton's method for finding the root to $f(x) = 0$.

32. (§4.2) Give an example (graphical or analytic) of a function, $f(x)$ for which Newton's Method:

1. Does not find a root for some choices of x_0.
2. Finds a root for every choice of x_0.
3. Falls into a cycle for some choice of x_0.
4. Converges slowly to the zero of $f(x)$.

33. (§4.2) How will Newton's Method perform for finding the root to $f(x) = \sqrt{|x|} = 0$?

34. (§4.2) Use Newton's Method to write a subroutine that, given z, computes $\sqrt[3]{z}$.

35. (§4.2) Use Newton's Method to devise a sequence $x_0, x_1, \ldots$ such that $x_k \to \ln 10$. Is this a reasonable way to write a subroutine that, given z, computes $\ln z$? (*Hint:* such a subroutine would require computation of e^{x_k}. Is this possible for rational x_k without using a logarithm? Is it practical?)

36. (§4.2) Letting $e_k = r - x_k$, derive the error relation

$$e_{k+1} = \frac{-f''(\xi_k)e_k^2}{2f'(x_k)},$$

by using Taylor's theorem. Where is ξ_k restricted to be?

37. (§4.2) Suppose you are using Newton's Method to find the root to the equation $\ln x = 0$. (We know the root is 1, but ignore that for the moment.) Using the error relation $e_{k+1} = -f''(\xi_k)e_k^2/2f'(x_k)$, can you find an interval (a, b) containing 1 such that if x_k is in this interval, successive iterates will converge to 1?

38. (§4.1–§4.3) Compare and contrast the Bisection, Newton, and Secant methods for finding the root to an equation $f(x) = 0$. What are their comparative advantages and disadvantages? Discuss convergence, applicability, computational issues, etc. Can you think of a function $g(x)$ that has a zero such that only one of these three methods is appropriate? Can you think of a function $h(x)$ that has a zero such that *none* of these methods is appropriate?