Exam 2 Preparation

The second midterm exam is Friday November 19, during the class period. *You must bring a blue book to the exam.* The exam covers §5.1,5.2,6.1,6.2,7.1,7.2,8.1–8.4 You should be prepared to answer at least the following questions:

1. (§5.1) Define the Lagrange Polynomials for a set of nodes $\{x_i\}_{i=0}^{n}$.

2. (§5.1) Find the Lagrange Polynomials for the nodes $x_0 = 1$, $x_1 = 2/3$, $x_2 = 4$.

3. (§5.1) Use Lagrange Polynomials to find the polynomial interpolant for the data

x	$\frac{1}{2}$	-2	5
y	1	-2	4

4. (§5.1) Use Newton's strategy to find the polynomial interpolant for the data from the previous problem. Leave the polynomial in 'nested' form.

5. (§5.1) Find the nested form of the polynomial interpolant of the data

x	1	3	4	6
y	-3	13	21	1

by completing the following divided differences table:

x	$f[\,]$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$
1	-3			
3	13			
4	21			
6	1			

6. (§5.1) We showed in class that there is a unique polynomial of degree $\leq n$ that interpolates a set of $n+1$ points $(x_i, f(x_i))$, where the x_i are distinct. Now consider the problem of *Hermite Interpolation* which also looks at derivatives of the interpolating polynomial:

(a) Find a polynomial $p(x)$ of lowest degree such that $p(1) = 3$, $p(2) = 5$, and $p'(1) = 1$. Do you think there is another polynomial of the same degree that has these properties?

(b) Make a conjecture regarding what degree polynomial is required to interpolate the set of data:

x	x_0	x_1	$\ldots$	x_n
$p(x)$	y_0	y_1	$\ldots$	y_n
$p'(x)$	y'_0	y'_1	$\ldots$	y'_n

(c) How might you show uniqueness of the Hermite interpolant?

7. (§5.2) State the polynomial interpolation error theorem (Theorem 5.8 of the notes).

8. (§5.2) How many equally spaced nodes are required to interpolate the function $\cos x$ to within 0.0001 on the interval $[0, \pi]$?

9. (§5.2) How many Chebyshev nodes are required to interpolate the function $\frac{1}{x}$ to within 0.000001 on the interval $[1, 2]$?

10. (§6.1) Give the definition of a linear spline.

11. (§6.1) Give the definition of the natural cubic spline.

12. (§6.1) Is the following function a linear spline on $[0, 4]$? Why or why not?

$$S(x) = \begin{cases} x^3 - 15 & : 0 \le x < 3 \\ 5x - 3 & : 3 \le x \le 4 \end{cases}$$

13. (§6.1) Is the following function a linear spline on $[0, 4]$? Why or why not?

$$S(x) = \begin{cases} x + 3 & : 0 \le x < 2 \\ -x + 6 & : 2 \le x \le 4 \end{cases}$$

14. (§6.1) Is the following function a quadratic spline on $[0, 4]$? Why or why not?

$$S(x) = \begin{cases} x^2 + 2x - 3 & : 0 \le x < 1 \\ 4x - 4 & : 1 \le x \le 4 \end{cases}$$

15. (§6.2) Find the natural cubic spline that interpolates the data

x	0	1	3
y	4	2	7

It may help to assume your answer has the form

$$S(x) = \begin{cases} Ax^3 + Bx^2 + Cx + 4 & : 0 \le x < 1 \\ D(x-1)^3 + E(x-1)^2 + F(x-1) + 2 & : 1 \le x \le 3 \end{cases}$$

16. (§7.1) Derive the approximation

$$f'(x) \approx \frac{4f(x+3h) + 5f(x) - 9f(x-2h)}{30h}$$

using Taylor's Theorem.

(a) What order approximation is this? (Assume $f(x)$ has bounded derivatives of arbitrary order.)

(b) Use this formula to approximate $f'(0)$, where $f(x) = x^4$, and $h = 0.1$

17. (§7.1) Assuming that $\phi(h) = Q + a_2h^2 + a_4h^4 + a_6h^6 \ldots$, find some combination of $\phi(h), \phi(h/4)$ which is a $\mathcal{O}\left(h^4\right)$ approximation to Q.

18. (§7.1) Suppose you have some great computational approximation to the quantity Q such that $\psi(h) = Q + a_3h^3 + a_6h^6 + a_9h^9 \ldots$ Can you find some combination of $\psi(h), \psi(h/2)$ which is a $\mathcal{O}\left(h^6\right)$ approximation to Q?

19. (§7.2) Give the recurrence formula for Richardson's Extrapolation. (Note: if you forget this you can derive it. Just remember that you want

$$D(n, m) = L + a_{2(m+1)} \left(\frac{h}{2^n}\right)^{2(m+1)} + a_{2(m+2)} \left(\frac{h}{2^n}\right)^{2(m+2)} + \ldots$$

and thus

$$D(n + 1, m) = L + 2^{-2(m+1)} a_{2(m+1)} \left(\frac{h}{2^n}\right)^{2(m+1)} + 2^{-2(m+2)} a_{2(m+2)} \left(\frac{h}{2^n}\right)^{2(m+2)} + \ldots$$

Thus you can combine $2^{2(m+1)} D(n + 1, m) + D(n, m)$ to get a $\mathcal{O}\left(h^{2(m+2)}\right)$ approximation to L.) (or you can just memorize the formula.)

20. (§7.2) Complete the following Richardson's Extrapolation Table, assuming the first column consists of values $D(n, 0)$ for $n = 0, 1, 2$:

$n \backslash m$	0	1	2
0	1		
1	0.25	?	
2	0.0625	?	?

21. (§8.2) Use the trapezoid rule, by hand to approximate

$$\int_0^1 \frac{4}{1 + x^2}\, \mathrm{d}x.$$

Use $n = 4$ subintervals. How good is your answer?

22. (§8.2) How many equal subintervals would be required to approximate

$$\int_0^1 \frac{4}{1 + x^2}\, \mathrm{d}x.$$

to within 0.0001? (*Hint:* Use the fact that $|f''(x)| \leq 8$ on $[0, 1]$ for $f(x) = 4/(1 + x^2)$)

23. (§8.3) *Simpson's Rule* for quadrature is given as

$$\int_a^b \approx \frac{\Delta x}{3} \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right],$$

where $\Delta x = (b - a)/n$, and n is assumed to be even. Show that Simpson's Rule for $n = 2$ is actually given by Romberg's Algorithm as $R(1, 1)$. As such we expect Simpson's Rule to be a $\mathcal{O}\left(h^4\right)$ approximation to the integral.

24. (§8.4) Consider the order 3 *Chebyshev Quadrature* rule:

$$\int_{-1}^1 f(x)\, \mathrm{d}x \approx c_3 \sum_{i=0}^{3} f(x_i)$$

Find the weighting c_3 and nodes x_i. For what order polynomials is this rule exact?

25. (§8.4) Determine a quadrature rule of the form

$$\int_0^1 f(x)\,\mathrm{d}x \approx Af(0) + Bf'(0) + Cf(1)$$

that is exact for polynomials of highest possible degree. What is the highest degree polynomial for which this rule is exact?

26. (§8.4) Find a quadrature rule of the form

$$\int_0^1 f(x)\,\mathrm{d}x \approx Af(0) + Bf(1/2) + Cf(1)$$

that is exact for polynomials of highest possible degree. What is the highest degree polynomial for which this rule is exact?

27. (§8.4) Find the Gaussian Quadrature rule with 2 nodes for the interval $[1, 5]$, *i.e.*, find a rule

$$\int_1^5 f(x)\,\mathrm{d}x \approx Af(x_0) + Bf(x_1)$$

Before you solve the problem, consider the following questions: do you expect the nodes to be the endpoints 1 and 5? do you expect the nodes to be arranged symmetrically around the midpoint of the interval?

28. (§8.4) Find the Gaussian Quadrature rule with 3 nodes for the interval $[-1, 1]$, *i.e.*, find a rule

$$\int_{-1}^1 f(x)\,\mathrm{d}x \approx Af(x_0) + Bf(x_1) + Cf(x_2)$$

To find the nodes x_0, x_1, x_2 you will have to find the zeroes of a cubic equation, which could be difficult. I think you will find, however, that the cubic polynomial has the midpoint of the interval, 0 as a zero, which reduces it to a quadratic.