Exam 1 Preparation

The first midterm exam is Monday October 18, during the class period. *You must bring a blue book to the exam.* Blue books are available from the bookstore. You also need to bring your student ID card or other form of ID (driver's license, passport, etc.)

You should be prepared to answer at least the following questions:

1. (§3.11) Approximate $\sqrt{4.1}$ using linearizations.

Answer: Let $f(x) = \sqrt{x}$. Then $f(x+h) \approx f(x) + f'(x)h$. Thus $\sqrt{4.1} = f(4+0.1) \approx f(4) + 0.1f'(4) = 2 + 0.1\,(1/4) = 2.025$

2. (§3.11) Approximate $\cos(44.8°)$ using linearizations.

Answer: Let $f(x) = \cos x$. Then $f(x+h) \approx f(x)+f'(x)h = \cos x - h\sin x$. Thus $\cos(44.8°) \approx \left(1/\sqrt{2}\right) + (-0.2°)\left(1/\sqrt{2}\right)$. We have to use radians, not degrees, for h, so this becomes $\cos(44.8°) \approx \left(1/\sqrt{2}\right) + (-0.2\pi/180)\left(1/\sqrt{2}\right)$, which you probably could not compute in your head.

3. (§3.11) Find the linearization of $\ln x$ at 1.

Answer: The linearization is $L(x) = \ln(1) + (x-1)\,\frac{\mathrm{d}}{\mathrm{d}x}\ln x\big|_{x=1} = x - 1$

4. (§3.11) A sphere is measured to have a radius of 10cm, with an error of ±1mm. *Estimate* the error in the calculated volume of the sphere.

Answer: We estimate the error using differentials. Let $V(r) = \frac{4\pi}{3}r^3$, be the calculated volume. Then the differential is $\mathrm{d}V(r) = 4\pi r^2\mathrm{d}r$. So our estimated error is $\mathrm{d}V(10) = 4\pi 10^2 0.1 = 40\pi$.

5. (§4.2) Verify that $f(x) = x^3 + x - 1$ satisfies the hypotheses of the Mean Value Theorem on $[0,2]$. Find a c in this interval that satisfies the conclusions of that theorem.

Answer: The function $f(x)$ is continuous and differentiable on the interval in question, thus the MVT applies. The MVT asserts the existence of a c such that $f'(c) = (f(2) - f(0))\,/\,(2-0) = (9 - -1)\,/2 = 5$. But $f'(c) = 3c^2 + 1$. Thus the desired c satisfies $3c^2 = 4$, so the answer is $c = \sqrt{4/3}$. Note the negative root is not in the interval $[0,2]$, so does not satisfy the conclusions of the MVT.

6. (§4.2) Can the equation $4 + 3x + x^7 = 0$ have *two* distinct roots. (*Hint:* assume there are two distinct roots, and show a contradiction.)

Answer: Suppose you have distinct a, b such that $f(a) = f(b)$. Because $f(x)$ is continuous and differentiable everywhere, we can apply the MVT (or, equivalently, Rolle's Theorem) to f on $[a,b]$. Thus there is a c such that $f'(c) = (0-0)\,/\,(b-a) = 0$. But $f'(c) = 7c^6 + 3$, which is clearly always strictly positive (*i.e.*, never negative or zero). Thus we have a contradiction to $f'(c) = 0$. Therefore the assumption that two distinct roots a, b exist was false.

7. (§4.2) Using the Mean Value Theorem, find upper and lower bounds on $f(4)$ for continuous, differentiable function $f(x)$ with the properties that $f(0) = 1$, and $-3 \leq f'(x) \leq 1$ for x in $[0,4]$.

Answer: MVT asserts that there is a c in $[0,4]$ such that $f'(c) = (f(4) - f(0))\,/\,(4-0) = (f(4)-1)\,/4$. Thus we directly have

$$-3 \leq \frac{f(4)-1}{4} \leq 1,$$

and thus, by manipulating these two inequalities, $-11 \leq f(4) \leq 5$.

8. (§4.2) Let $f(x) = |x-2|$. Show there is no c such that $f(4) - f(1) = f'(c)(4-1)$. Doesn't this contradict the Mean Value Theorem?

Answer: First note that $(f(4) - f(1))/(4-1) = (|2| - |-1|)/3 = 1/3$. However, we have

$$f'(x) = \begin{cases} 1 & \text{if } x > 2, \\ -1 & \text{if } x < 2. \end{cases}$$

Thus there is no such c. This does not contradict the MVT because the function $f(x)$ is not differentiable on $[1,4]$. In particular, its derivative is not defined at 2.

9. (§4.10) Find the most general antiderivative of the following:

$$f(x) = x^3 - 2, \qquad g(x) = \sqrt{2x} + x^{-4}, \qquad h(x) = 4\cos x - \sin x,$$
$$j(x) = 1 + x^{3/5}, \qquad k(x) = 4/(1+x^2)$$

Answer:

$$\int f(x) = x^4/4 - 2x + C, \qquad \int g(x) = (2x)^{3/2}/3 - x^{-3}/3 + C,$$
$$\int h(x) = 4\sin x + \cos x + C, \qquad \int j(x) = x + \frac{5}{8}x^{8/5} + C, \qquad \int k(x) = 4\arctan x + C$$

10. (§5.2) Estimate $\int_0^1 x^2 \mathrm{d}x$ by a Riemann sum, using 4 subintervals, and using the right hand endpoint rule. Is this an underestimate or overestimate of the area?

11. (§5.3) State the Fundamental Theorem of Calculus, parts 1 and 2.

12. (§5.3) Find the derivative of $f(x) = \int_1^x \sqrt{1+t^3}\mathrm{d}t$.

Answer: $f'(x) = \sqrt{1+x^3}$ by the Fundamental Theorem of Calculus, part 1 (FTC I).

13. (§5.3) Find the derivative of $f(x) = \int_x^3 \sin\left(\sqrt[4]{t}\right) \mathrm{d}t$.

Answer: First write $f(x) = -\int_3^x \sin\left(\sqrt[4]{t}\right) \mathrm{d}t$, then use FTC I to get $f'(x) = -\sin\left(\sqrt[4]{x}\right)$.

14. (§5.3) Find the derivative of $f(x) = \int_0^{x^2} \sqrt{1+t^4}\mathrm{d}t$.

Answer: First write $f(x) = h(x^2)$, where $h(x) = \int_0^x \sqrt{1+t^4}\mathrm{d}t$. By the chain rule, $f'(x) = h'(x^2)2x$. By FTC I, $h'(x) = \sqrt{1+x^4}$. Thus $f'(x) = 2x\sqrt{1+x^8}$.

15. (§5.3) Let $f(x) = \int_{1/x}^{1+x^2} t^3 \mathrm{d}t$. Evaluate $f'(2)$. (*Hint:* split the integral as $f(x) = \int_{1/x}^1 t^3 \mathrm{d}t + \int_1^{1+x^2} t^3 \mathrm{d}t$, and use the Fundamental Theorem twice.)

16. (§5.3) Give a physical interpretation of the definite integral $\int_1^2 x^2 \mathrm{d}x$. Make a drawing (graph) to show this interpretation.

Answer: The physical interpretation is the area under the graph of $y = x^2$, but above the x-axis, and between the lines $x = 1$ and $x = 2$.

17. (§5.3) Evaluate the following definite integrals:

$$\int_2^4 \mathrm{d}x \quad \int_0^1 6x^2 \mathrm{d}x \quad \int_\pi^{2\pi} \cos\theta \mathrm{d}\theta \quad \int_0^1 \left(3 + x\sqrt{x}\right) \mathrm{d}x \quad \int_1^2 \frac{3+x^2}{x^3}\mathrm{d}x \quad \int_1^e t^{-1}\mathrm{d}t$$

Answer:

$$\int_2^4 \mathrm{d}x = x\Big|_2^4 = 4-2 = 2 \qquad \int_0^1 6x^2\mathrm{d}x = 6x^3/3\Big|_0^1 = 2(1-0) = 2$$

$$\int_\pi^{2\pi} \cos\theta\mathrm{d}\theta = \sin\theta\Big|_\pi^{2\pi} = 0-0 = 0$$

$$\int_0^1 \left(3+x\sqrt{x}\right)\mathrm{d}x = 3x + \frac{2}{5}x^{5/2}\Big|_0^1 = \left(3+\frac{2}{5}1\right) - (0+0) = 3+\frac{2}{5}$$

$$\int_1^2 \frac{3+x^2}{x^3}\mathrm{d}x = \int_1^2 \frac{3}{x^3}+\frac{1}{x}\mathrm{d}x = -\frac{3}{2}x^{-2}+\ln x\Big|_1^2 = \left(-\frac{3}{2\,2^2}+\ln 2\right) - \left(-\frac{3}{2\,1^2}+\ln 1\right) = \frac{9}{8}+\ln 2.$$

$$\int_1^e t^{-1}\mathrm{d}t = \ln t\Big|_1^e = \ln e - \ln 1 = 1.$$

18. (§5.4) Evaluate the indefinite integrals: (Don't forget the "$+C$")

$$\int x^n\mathrm{d}x,\ \text{with}\ n\neq -1, \quad \int x^{-1}\mathrm{d}x, \quad \int e^x\mathrm{d}x, \quad \int \sin x\mathrm{d}x, \quad \int \cos x\mathrm{d}x,$$

$$\int 1/(1+x^2)\mathrm{d}x, \quad \int \sec^2 x\mathrm{d}x, \quad \int \sec x\tan x\mathrm{d}x,$$

Answer: You need to have these memorized.

19. (§5.5) Evaluate the indefinite and definite integrals:

$$\int x^2\left(x^3-1\right)^6\mathrm{d}x, \quad \int \frac{x+5x^4}{\sqrt{1+x^2+2x^5}}\mathrm{d}x, \quad \int \frac{\sin x}{1+\cos^2 x}\mathrm{d}x, \quad \int \frac{1}{x\ln x}\mathrm{d}x,$$

$$\int \sqrt{4+3x}\mathrm{d}x, \quad \int e^x\sin\left(1+e^x\right)\mathrm{d}x, \quad \int \frac{e^{1/x}}{x^2}\mathrm{d}x,$$

$$\int_0^2 (x-1)^4\mathrm{d}x, \quad \int_0^{\sqrt{\pi}} x\cos x^2\mathrm{d}x, \quad \int_0^\pi \sin x\sin\left(\cos x\right)\mathrm{d}x, \quad \int_1^{e^\pi} \frac{\cos\left(\ln x\right)}{x}\mathrm{d}x,$$

Answer: These all use u-substitution. I only answer a few here

$$\int x^2\left(x^3-1\right)^6 \mathrm{d}x = \frac{1}{3}\int u^6 \mathrm{d}u = \frac{1}{21}\left(x^3-1\right)^7 + C$$
$$\left[u = x^3 - 1 \Rightarrow \mathrm{d}u = 3x^2 \mathrm{d}x\right]$$
$$\int \frac{\sin x}{1+\cos^2 x}\mathrm{d}x = -\int \frac{1}{1+u^2}\mathrm{d}u = -\arctan\cos x + C$$
$$[u = \cos x \Rightarrow \mathrm{d}u = -\sin x \mathrm{d}x]$$
$$\int \frac{1}{x\ln x}\mathrm{d}x = \int \frac{1}{u}\mathrm{d}u = \ln(\ln x) + C$$
$$[u = \ln x \Rightarrow \mathrm{d}u = (1/x)\,\mathrm{d}x]$$
$$\int \frac{e^{1/x}}{x^2}\mathrm{d}x = -\int e^u \mathrm{d}u = -e^{1/x} + C$$
$$\left[u = 1/x \Rightarrow \mathrm{d}u = -\left(1/x^2\right)\mathrm{d}x\right]$$
$$\int_0^{\sqrt{\pi}} x\cos x^2 \mathrm{d}x = \frac{1}{2}\int_0^{\pi} \cos u \mathrm{d}u = \frac{1}{2}\ \sin u\Big|_0^{\pi} = 0$$
$$\left[u = x^2 \Rightarrow \mathrm{d}u = 2x\mathrm{d}x\right]$$
$$\int_1^{e^\pi} \frac{\cos(\ln x)}{x}\mathrm{d}x = \int_0^{\pi} \cos u \mathrm{d}u = \ \sin u\Big|_0^{\pi} = 0$$
$$[u = \ln x \Rightarrow \mathrm{d}u = (1/x)\,\mathrm{d}x]$$

20. (§5.6) Give the definition of $\ln x$.
Answer: We defined

$$\ln x = \int_1^x \frac{1}{t}\mathrm{d}t.$$

21. (§5.6) Use the fact that for $t > 1$,

$$\frac{1}{t^2} \leq \frac{1}{t} \leq \frac{1}{\sqrt{t}}$$

to estimate $\ln 2$.
Answer: Using the definition of the natural log, and facts about integrals we get

$$\int_1^2 \frac{1}{t^2}\mathrm{d}t \leq \ln 2 = \int_1^2 \frac{1}{t}\mathrm{d}t \leq \int_1^2 \frac{1}{\sqrt{t}}\mathrm{d}t.$$

This gives $\frac{1}{2} \leq \ln 2 \leq 2\left(\sqrt{2}-1\right) \approx 0.83$.

22. (§6.1) Find the area of each of the regions enclosed by the given curves:

1. $y = 4x,\ y = 1/x,\ x = 1,\ x = 4.$
2. $y = x + 2,\ y = 4 - x^2.$
3. $y = 1/x,\ y = 1/x^2,\ x = 3.$
4. $y = x^2,\ x = y^2.$
5. $y = \sin x,\ y = \sin 2x,\ x = 0,\ x = \pi/2$. (*Hint:* you may wish to use the fact that $\sin 2x = 2 \sin x \cos x$.)

Answer: 1. Because $4x > 1/x$ on the interval in question, the area of the region is $\int_1^4 4x - (1/x)\,\mathrm{d}x = 2x^2 - \ln x\Big|_1^4 = (32 - \ln 4) - (2 - 0) = 30 - 2\ln 2.$

2. First set the two functions equal to each other to find their intercepts: $x + 2 = 4 - x^2 \Leftrightarrow x^2 + x - 2 = 0 \Leftrightarrow (x+2)(x-1) = 0$. Thus the two curves intersect when $x = 1$ and $x = -2$. In the interval $(-2, 1)$, I know that $4 - x^2 > x + 2$. I figured this out by checking at $x = 0$, but you could just graph the two curves. The area between them is then

$$\int_{-2}^{1} 4 - x^2 - (x+2)\mathrm{d}x = 2x - x^3/3 - x^2/2\Big|_{-2}^{1} = 9/2$$

3. The functions $y = 1/x,\ y = 1/x^2$ intersect at $x = 1$. Over the interval $(1, 3)$, $1/x > 1/x^2$, so the area is

$$\int_1^3 1/x - 1/x^2\mathrm{d}x = \int_1^3 x^{-1} - x^{-2}\mathrm{d}x = \ln x + x^{-1}\Big|_1^3 = (\ln 3 + 1/3) - (\ln 1 + 1) = (\ln 3) - 2/3.$$

4. We can write these as $y = x^2$ and $y = \sqrt{x}$, which intersect only at $x = 0$, and $x = 1$. In the interval $(0, 1)$, it is the case that $\sqrt{x} > x^2$, and thus the area is

$$\int_0^1 \sqrt{x} - x^2\mathrm{d}x = 2/3x^{3/2} - x^3/3\Big|_0^1 = (2/3 - 1/3) - (0 - 0) = 1/3.$$

5. Because $\sin 2x = 2 \sin x \cos x$, then

$$\sin x = \sin 2x \Leftrightarrow \sin x = 2 \sin x \cos x \Leftrightarrow \sin x = 0 \quad \text{or} \quad 1/2 = \cos x.$$

Thus the curves intersect at $x = 0$ and $x = \pi/3 = 60°$. Thus we have to find the area by splitting the integral in two:

$$\begin{aligned}\int_0^{\pi/2} |\sin x - \sin 2x|\,\mathrm{d}x &= \int_0^{\pi/3} (\sin 2x - \sin x)\,\mathrm{d}x + \int_{\pi/3}^{\pi/2} (\sin x - \sin 2x)\,\mathrm{d}x \\ &= -(\cos 2x)/2 + \cos x\Big|_0^{\pi/3} + -\cos x + (\cos 2x)/2\Big|_{\pi/3}^{\pi/2} \\ &= [(1/4 + 1/2) - (-1/2 + 1)] + [(-0 + -1/2) - (-1/2 - 1/4)] = 1/2\end{aligned}$$

23. (§6.2) Let $A(x)$ be the cross sectional area of shape S, which is bounded by the planes $x = a$, $x = b$. What is the volume of S?

Answer: You need to know this one:

$$V = \int_a^b A(x)\mathrm{d}x.$$

24. (§6.2) Find the volume of each of the following shapes:

1. Rotation of region bounded by $y = \sqrt{x}$, $x = 0$, $x = 2$ around the x-axis.
2. Rotation of region bounded by $y = \sqrt{\ln x^{1/x}}$, $x = 1$, $x = e$ around the x-axis.
3. Rotation of region bounded by $y = x^2$, $y = 0$, $y = 4$ around the y-axis.
4. Rotation of region bounded by $y = x^2$, $x = 0$, $x = 2$ around the y-axis.
5. Rotation of region bounded by $y = x^2$, $y = 4$, around the line $y = 4$.
6. Rotation of region bounded by $y = x$, $y = \sqrt{x}$, around the line $x = 2$.

Answer: 1. We get

$$V = \int_0^2 \pi\left(\sqrt{x}\right)^2 \mathrm{d}x = \pi x^2/2\Big|_0^2 = 2\pi.$$

2. The volume is

$$V = \int_1^e \pi\left(\sqrt{\ln x^{1/x}}\right)^2 \mathrm{d}x = \int_1^e \pi \ln x^{1/x}\mathrm{d}x = \pi \int_1^e \frac{1}{x}\ln x\mathrm{d}x\pi \int_0^1 u\mathrm{d}u = \pi\ u^2/2\Big|_0^1 = \pi/2.$$

3. We have

$$V = \int_0^4 \pi y\mathrm{d}y = \pi\ y^2/2\Big|_0^4 = 8.$$

4. Not answered

5.

$$V = \int_{-2}^2 \pi\left(4 - x^2\right)^2 \mathrm{d}x = \pi \int_{-2}^2 16 - 8x^2 + x^4\mathrm{d}x = \pi\ 16x - (8/3)x^3 + x^5/5\Big|_{-2}^2 = 512\pi/15.$$

25. (§6.5) Find the average value of the function on the given interval:

1. $f(x) = \sqrt{x}, \quad [0, 2]$.
2. $f(x) = x\cos\left(x^2/2\right), \quad [0, \sqrt{\pi}]$.

Answer: 1. $f_{avg} = \frac{1}{2-0}\int_0^2 \sqrt{x}\mathrm{d}x = \frac{1}{3}\ x^{3/2}\Big|_0^2 = 2^{3/2}/3.$

2. $f_{avg} = \frac{1}{\sqrt{\pi}-0}\int_0^{\sqrt{\pi}} x\cos\left(x^2/2\right)\mathrm{d}x = \pi^{-1/2}\int_0^{\pi/2}\cos u\mathrm{d}u = \pi^{-1/2}\ \sin u\Big|_0^{\pi/2} = \pi^{-1/2}.$

26. (misc.) Give an upper bound on $\int_0^1 \sin\sqrt[4]{t}\mathrm{d}t$. Give a lower bound on this integral using the fact that $\sin x \geq x$ for x in $[0, 1]$.