Exam 2 Preparation

The second midterm exam is Monday November 15, during the class period. *You must bring a blue book to the exam.* Blue books are available from the bookstore. You also need to bring your student ID card or other form of ID (driver's license, passport, etc.)

The following formulæ will be provided on your exam:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2},$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

Everything else must be committed to memory.

The exam concentrates on §7.1-7.5, 7.7, 10.3, Appendix G, Supplement §1, 2, 3, 4. You should not, of course, forget any of the older material, like integration by substitution, etc. You should be prepared to answer questions like those that follow. Note that I have not checked many of these problems, some of them may be unsolveable, impractically difficult or generally insane. You are warned.

1. (§7.1) Integration by parts: (a) $\int xe^x\,dx$. (b) $\int x^2e^x\,dx$. (c) $\int x\sin x\,dx$. (d) $\int x\ln x\,dx$. (e) $\int \arctan 4x\;dx$. (f) $\int_0^1 (x^2{+}1)e^{-x}\,dx$. (g) $\int_1^2 x^{-2}\ln x\,dx$. (h) $\int \sin x\ln(\cos x)\;dx$. (i) $\int \cos(\ln x)\;dx$.

2. (§10.3) Polar Coordinates: (a) Convert the point $(1,2)$ to polar coordinates. (b) Convert the point $(-2,0)$ to polar coordinates. (c) Convert the point $(2,\pi/3)$ to Cartesian coordinates. (d) Convert the point $(1.5, 5\pi/4)$ to Cartesian coordinates. (e) Graph the curve given by the polar equation $r=2$. (f) Graph the curve given by the polar equation $r=\sin\theta$. (g) Graph the curve given by the polar equation $r=\sin 2\theta$.

3. (Appendix G) Arithmetic with Complex numbers. Evaluate the following:
- (a) $(4+3i)+(2-5i)$.
- (b) $(1-3i)(3-5i)$.
- (c) $(5-2i)-(3-i)$.
- (d) $(-1+i)\,/\,(5-i)$
- (e) $(12(\cos 8\pi/5 + i\sin 8\pi/5))\,/\,(3(\cos\pi/10 + i\sin\pi/10))$
- (f) $3(\cos\pi/3 + i\sin\pi/3)2(\cos\pi/2 + i\sin\pi/2)$.
- (g) $[2(\cos 2\pi/3 + i\sin 2\pi/3)]^4$.
- (h) $[\cos(-\pi/6) + i\sin(-\pi/6)]^8$.
- (i) $[3(\cos\pi/3 + i\sin\pi/3)]^3$.

4. (Appendix G) Complex numbers: modulus, argument, conjugate: (a) Convert $2+i$ to polar form, *i.e.*, find r,θ such that $2+i = r(\cos\theta + i\sin\theta)$. (b) Convert $-1-i$ to polar form. (c) Convert $3(\cos 7\pi/6 + i\sin 7\pi/6)$ to $a+bi$ form. (d) What is $|z|$ when $z=3+4i$? (e) What is $\overline{z}$ when $z=2+10i$? (f) What is $\overline{4}$? (g) Give the four complex fourth roots of -2. (h) Give the three complex cube roots of 8. (i) Find the five zeroes to the polynomial x^5+32.

5. (Supplement §1) Euler's Formula:
1. Evaluate $e^{3+2i\pi}$.
2. Evaluate $e^{1+2i}e^{1-2i}$.

6. (§7.2) Questions of the form $\int \cos^m x \sin^n x \, dx$. There are two main cases:

(I) If one of n or m is odd, then keep one power of the appropriate cos or sin, and convert the rest using $\cos^2 x + \sin^2 x = 1$. Then make a u substitution.
For example you should convert $\int \cos^5 x \sin^2 x \, dx$ to $\int \cos x \left(1 - \sin^2 x\right)^2 \sin^2 x \, dx$ and use $u = \sin x$.

(II) If both n and m are even, then use the identities $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$. to lower the powers.

Questions of this kind: (a) $\int \cos^3 x \sin^1 x \, dx$. (b) $\int \cos^2 x \sin^3 x \, dx$. (c) $\int \sec^6 x \sin^5 x \, dx$. (d) $\int \cos^2 x \sin^4 x \, dx$. (e) $\int_0^{\pi/2} \cos^5 x \, dx$. (f) $\int \sin^3 x \, dx$. (g) $\int \frac{\sin^3(\ln x)}{x} \, dx$. (h) $\int \sin^4 x \, dx$. (i) $\int \cos^4 x \, dx$. (j) $\int \sin^8 x \, dx$.

7. (§7.2) Questions of the form $\int \tan^m x \sec^n x \, dx$. There are three cases:

(I) If m is odd, keep a $\sec x \tan x$, and convert $\tan^{m-1} x$ to powers of $\sec x$ using $\tan^2 x + 1 = \sec^2 x$. Use the u substitution $u = \sec x$.

(II) If n is *even*, keep a $\sec^2 x$, and convert $\sec^{n-2} x$ to powers of $\tan x$ using $\tan^2 x + 1 = \sec^2 x$. Use the u substitution $u = \tan x$.

(III) If m is even and n is odd, protest and complain! This case is difficult. One thing you can do is keep a $\sec x$, and convert $\cos^{1-n} x$ to a denominator in $\sin x$ using $\sin^2 x + \cos^2 x = 1$, use $u = \sin x$, then try a partial fraction expansion. Not very pretty.

Questions of this kind: (a) $\int \tan^3 x \sec x \, dx$. (b) $\int \tan^3 x \sec^3 x \, dx$. (c) $\int \tan^2 x \sec^4 x \, dx$. (d) $\int \tan^3 x \sec^6 x \, dx$. (e) $\int \tan^2 x \, dx$. (f) $\int \sec^3 x \, dx$.

8. (§7.3) 'Hidden' Trig integrals. In these there is usually a component that looks like Pythagora's Theorem, *e.g.*, a $\sqrt{x^2+9}$ or a $\sqrt{25-x^2}$. Draw a triangle with the right sides, make θ one of the angles and express the integral in terms of θ. (a) $\int \sqrt{100-x^2} \, dx$. (b) $\int \frac{dx}{\sqrt{9+x^2}}$. (c) $\int \frac{x \, dx}{\sqrt{9+x^2}}$. (d) $\int \frac{10 \, dx}{x(100+x^2)^{3/2}}$. (e) $\int_0^2 x^3\sqrt{x^2+4} \, dx$. (f) $\int \frac{\sqrt{1+x^2} \, dx}{x}$.

9. (§7.3) Sometimes the Hidden Trig integrals are obscured. In these you need to make a substitution to get it to the right form: (a) $\int \frac{x-2}{x^2-2x+5} \, dx$. (Note that the denominator is irreducible, so a Partial Fraction Expansion will do no good here. Instead, try $u = x - 2$.) (b) $\int \frac{dx}{(x+3)\sqrt{x^2+6x-25}}$. (c) $\int_{-3}^{5} \frac{dx}{(55-6x-x^2)^{3/2}}$.

10. (§7.4) Partial Fraction Expansion, *i.e.*, questions of the form $\int \frac{p(x)}{q(x)} \, dx$ for polynomials p, q. There is a three step strategy:

(I) If $\deg(p) \geq \deg(q)$, then do a 'long division with remainder' to find polynomials s, r such that $p(x)/q(x) = s(x) + r(x)/q(x)$, and $\deg r < \deg q$. Since s can be integrated easily, we need not look at it anymore.

(II) Express $q(x)$ as linear and irreducible quadratic factors. (In this class, the factoring will be either trivial *e.g.*, $q(x) = x^3 - 9x$, or q will be given to you factored.)

(III) Fraction magic! Remember that a linear factor $ax + b$ corresponds to a summand $\frac{K}{ax+b}$, while an irreducible quadratic factor $ax^2 + bx + c$ corresponds to a summand $\frac{Kx+L}{ax^2+bx+c}$. Repeated factors are repeated in the sum.

Examples:

$$\frac{x^2+3x-2}{(x-1)x(x+4)} \Rightarrow \frac{A}{x-1}+\frac{B}{x}+\frac{C}{x+4}$$
$$\frac{2x^2-3}{(x+1)^3(2x-3)} \Rightarrow \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}+\frac{D}{2x-3}$$
$$\frac{x^4+3x-2}{(x^2+1)(3x+7)(x^2+3x+4)} \Rightarrow \frac{Ax+B}{x^2+1}+\frac{C}{3x+7}+\frac{Dx+E}{x^2+3x+4}$$
$$\frac{x^5-3x^4-2x}{x(x^2+2)^2(x-2)(x+1)^2} \Rightarrow \frac{A}{x}+\frac{Bx+C}{x^2+2}+\frac{Dx+E}{(x^2+2)^2}+\frac{F}{x-2}+\frac{G}{x+1}+\frac{H}{(x+1)^2}.$$

Questions of this kind: (a) $\int \frac{x^4+2x^3-3}{x(x-1)(x+3)}\,\mathrm{d}x$. (b) $\int \frac{x^3-2x^3-3}{(x-2)(x+1)^2(2x+1)}\,\mathrm{d}x$. (c) $\int \frac{2x^2-x+1}{(x^2+3)(x-1)}\,\mathrm{d}x$. (d) $\int \frac{-x^2+10x-1}{(x^2+8)(x+1)^2}\,\mathrm{d}x$. (e) $\int \frac{-x^3-2x}{(x^2+1)^2(x-1)^2}\,\mathrm{d}x$. (f) $\int \frac{x^2}{(x+1)^3}\,\mathrm{d}x$. (g) $\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,\mathrm{d}x$. (h) $\int_0^1 \frac{x}{x^2+4x+13}\,\mathrm{d}x$. (i) $\int \frac{x^2}{x^2+1}\,\mathrm{d}x$. (j) $\int \frac{\mathrm{d}x}{x^4+x^2}\,\mathrm{d}x$. (k) $\int \frac{x^2+3x-2}{(x-1)x(x+4)}\,\mathrm{d}x$. (l) $\int \frac{2x^2-3}{(x+1)^3(2x-3)}\,\mathrm{d}x$.

11. (§7.4) Partial Fraction Expansion may also be obscured. Consider the example

$$\int \frac{\mathrm{d}x}{\sqrt[5]{x}+\sqrt[3]{x}}.$$

In this case if you let $u^{15}=x$, you can rewrite this integral as

$$\int \frac{15u^{14}\,\mathrm{d}u}{u^3+u^5}.$$

which is a rational function.
Questions of this kind: (a) $\int \frac{e^{3x}-e^{2x}+6e^x}{e^{3x}-e^x}\,\mathrm{d}x$. (b) $\int \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x$.

12. (§7.5) In reality it may not be immediately obvious how to solve an integral, as they will not be given to you with the relevant section numbers. Thus you should look over the strategies given in §7.5 to approach integrals.
(a) $\int \frac{e^{2x}+4e^x}{e^{2x}+2e^x+5}\,\mathrm{d}x$. (b) $\int \sqrt{3-2x-x^2}\,\mathrm{d}x$. (c) $\int_0^{\pi/4} \tan^5 x \sec^3 x\,\mathrm{d}x$. (d) $\int \arctan x\,\mathrm{d}x$. (e) $\int \frac{x^4}{x^{10}+16}\,\mathrm{d}x$. (f) $\int \frac{1+e^x}{1-e^x}\,\mathrm{d}x$. (g) $\int x\sin x\,\mathrm{d}x$. (h) $\int \left(1+\sqrt{x}\right)^3\,\mathrm{d}x$. (i) $\int x^2\sin x\,\mathrm{d}x$. (j) $\int x\sin^2 x\,\mathrm{d}x$. (k) $\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$. (l) $\int \frac{x}{x^4+4x^2+3}\,\mathrm{d}x$. (m) $\int \ln\left|\frac{x^3-x}{x+1}\right|\,\mathrm{d}x$. (n) $\int_{-1}^1 x^{-3}\,\mathrm{d}x$. (o) $\int (\ln x)^2\,\mathrm{d}x$. (p) $\int e^x\cos^3 e^x\,\mathrm{d}x$. (q) $\int \frac{\mathrm{d}x}{x+\sqrt[3]{x}}\,\mathrm{d}x$. (r) $\int \sin\sqrt{x}\,\mathrm{d}x$.

13. (§7.7) Approximate Integration. (a) Approximate the definite integral $\int_0^2 \frac{1}{1+x^2}\,\mathrm{d}x$. using the Trapezoidal Rule, the Midpoint Rule, and Simpson's Rule, with $n=4$. (Compare to the real answer $\arctan 2 \approx 1.1071$.) (b) Approximate the definite integral $\int_0^1 e^{x^2}\,\mathrm{d}x$. using the Trapezoidal Rule, the Midpoint Rule, and Simpson's Rule, with $n=6$.