Exam 2 Preparation with Selected Answers

The second midterm exam is Monday November 15, during the class period. *You must bring a blue book to the exam.* Blue books are available from the bookstore. You also need to bring your student ID card or other form of ID (driver's license, passport, etc.)

The following formulæ will be provided on your exam:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2},$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

Everything else must be committed to memory.

The exam concentrates on §7.1-7.5, 7.7, 10.3, Appendix G, Supplement §1, 2, 3, 4. You should not, of course, forget any of the older material, like integration by substitution, etc. You should be prepared to answer questions like those that follow. Note that I have not checked many of these problems, some of them may be unsolveable, impractically difficult or generally insane. You are warned.

1. (§7.1) Integration by parts: (a) $\int xe^x\,\mathrm{d}x$. (b) $\int x^2e^x\,\mathrm{d}x$. (c) $\int x\sin x\,\mathrm{d}x$. (d) $\int x\ln x\,\mathrm{d}x$. (e) $\int \arctan 4x\,\mathrm{d}x$. (f) $\int_0^1 (x^2+1)e^{-x}\,\mathrm{d}x$. (g) $\int_1^2 x^{-2}\ln x\,\mathrm{d}x$. (h) $\int \sin x \ln(\cos x)\,\mathrm{d}x$. (i) $\int \cos(\ln x)\,\mathrm{d}x$.

Answer: (a) use $u = x$, $\mathrm{d}v = e^x\,\mathrm{d}x$, to get $xe^x - e^x + C$. (b) use $u = x^2$, $\mathrm{d}v = e^x\,\mathrm{d}x$, to get $x^2e^x - 2[xe^x - e^x] + C$. (c) use $u = x$, $\mathrm{d}v = \sin x\,\mathrm{d}x$, to get $-x\cos x + \sin x + C$. (d) use $u = \ln x$, $\mathrm{d}v = x\,\mathrm{d}x$. (e) use $u = \arctan 4x$, $\mathrm{d}v = \mathrm{d}x$. (f) expand to use two integrals, the former uses a previous problem. (g) use $u = \ln x$, $\mathrm{d}v = x^{-2}\,\mathrm{d}x$. (h) not answered yet. (i) not answered yet.

2. (§10.3) Polar Coordinates: (a) Convert the point $(1, 2)$ to polar coordinates. (b) Convert the point $(-2, 0)$ to polar coordinates. (c) Convert the point $(2, \pi/3)$ to Cartesian coordinates. (d) Convert the point $(1.5, 5\pi/4)$ to Cartesian coordinates. (e) Graph the curve given by the polar equation $r = 2$. (f) Graph the curve given by the polar equation $r = \sin\theta$. (g) Graph the curve given by the polar equation $r = \sin 2\theta$.

3. (Appendix G) Arithmetic with Complex numbers. Evaluate the following:

(a) $(4 + 3i) + (2 - 5i)$.
(b) $(1 - 3i)(3 - 5i)$.
(c) $(5 - 2i) - (3 - i)$.
(d) $(-1 + i) / (5 - i)$
(e) $(12(\cos 8\pi/5 + i\sin 8\pi/5)) / (3(\cos \pi/10 + i\sin \pi/10))$
(f) $3(\cos \pi/3 + i\sin \pi/3)2(\cos \pi/2 + i\sin \pi/2)$.
(g) $[2(\cos 2\pi/3 + i\sin 2\pi/3)]^4$.
(h) $[\cos(-\pi/6) + i\sin(-\pi/6)]^8$.
(i) $[3(\cos \pi/3 + i\sin \pi/3)]^3$.

Answer: (a) $6 - 2i$.
(b) $-14 - 14i$.
(c) $2 - i$.
(d) not answered yet.

(e) $4(\cos 3\pi/2 + i \sin 3\pi/2)$.
(f) $6(\cos 5\pi/6 + i \sin 5\pi/6)$.
(g) $16(\cos 8\pi/3 + i \sin 8\pi/3)$.
(h) not answered yet.
(i) -27.

4. (Appendix G) Complex numbers: modulus, argument, conjugate: (a) Convert $2+i$ to polar form, *i.e.*, find r, θ such that $2 + i = r(\cos\theta + i\sin\theta)$. (b) Convert $-1-i$ to polar form. (c) Convert $3(\cos 7\pi/6 + i \sin 7\pi/6)$ to $a+bi$ form. (d) What is $|z|$ when $z = 3+4i$? (e) What is $\overline{z}$ when $z = 2+10i$? (f) What is $\overline{4}$? (g) Give the four complex fourth roots of -2. (h) Give the three complex cube roots of 8. (i) Find the five zeroes to the polynomial $x^5 + 32$.

Answer: (a) $2+i = \sqrt{5}\left(\cos\arctan\frac{1}{2} + i\sin\arctan\frac{1}{2}\right)$ (b) $-1-i = \sqrt{2}\left(\cos 5\pi/4 + i\sin 5\pi/4\right)$. (c) Convert $3(\cos 7\pi/6 + i \sin 7\pi/6)$ to $a+bi$ form. (d) 5. (e) $\overline{z} = 2-10i$. (f) $\overline{4} = 4$. (g) not answered yet. (h) $2, 2(\cos 2\pi/3 + i\sin 2\pi/3), 2(\cos 4\pi/3 + i \sin 4\pi/3)$ (i) the five complex fifth roots of -32.

5. (Supplement §1) Euler's Formula:

1. Evaluate $e^{3+2i\pi}$.

Answer: $e^3(\cos 2\pi + i\sin 2\pi) = e^3$.

1. Evaluate $e^{1+2i}e^{1-2i}$.

Answer: $e^{1+2i}e^{1-2i} = e^2$.

6. (§7.2) Questions of the form $\int \cos^m x \sin^n x\, dx$. There are two main cases:

(I) If one of n or m is odd, then keep one power of the appropriate cos or sin, and convert the rest using $\cos^2 x + \sin^2 x = 1$. Then make a u substitution.
For example you should convert $\int \cos^5 x \sin^2 x\, dx$ to $\int \cos x \left(1 - \sin^2 x\right)^2 \sin^2 x\, dx$ and use $u = \sin x$.

(II) If both n and m are even, then use the identities $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$. to lower the powers.

Questions of this kind: (a) $\int \cos^3 x \sin^1 x\, dx$. (b) $\int \cos^2 x \sin^3 x\, dx$. (c) $\int \sec^6 x \sin^5 x\, dx$. (d) $\int \cos^2 x \sin^4 x\, dx$. (e) $\int_0^{\pi/2} \cos^5 x\, dx$. (f) $\int \sin^3 x\, dx$. (g) $\int \frac{\sin^3(\ln x)}{x}\, dx$. (h) $\int \sin^4 x\, dx$. (i) $\int \cos^4 x\, dx$. (j) $\int \sin^8 x\, dx$.

Answer: (a) use $u = \sin x$ to get $\cos^4 x/4 + C$. (b) convert to $\int \cos^2 x(1-\cos^2 x)\sin x\, dx$, use $u = \cos x$. (c) convert to $\int \sec^6 x(1-\cos^2 x)^2 \sin x\, dx$, use $u = \cos x$. (d) terrible. not answered yet. (e) convert to $\int_0^{\pi/2}(1-\sin^2 x)^2 \cos x\, dx$, or or $\int_0^{\pi/2}(1-2\sin^2 x+\sin^4 x)\cos x\, dx$. use $u = \sin x$ to get $\int_0^1 (1-2u^2+u^4)\, du = u - 2u^3/3 + u^5/5\Big|_0^1 = 1 - 2/3 + 1/5 = 8/15$ (f) convert to $\int (1-\cos^2 x)\sin x\, dx$, use $u = \cos x$. (g) first substitute $u = \ln x$ to convert this to $\int \sin^3 u\, du$. then apply the previous problem. (h) use the double angle formula to convert to $\int 0.25(1-\cos 2x)^2\, dx$. expand to $0.25\int 1 - 2\cos 2x + \cos^2 2x\, dx$. use the double angle formula for cosine to convert $\cos^2 2x$. (i) use the double angle formula to convert to $\int 0.25(1+\cos 2x)^2\, dx$. expand to $0.25\int 1 + 2\cos 2x + \cos^2 2x\, dx$. now convert $\cos^2 2x$ using the double angle formula for cosine. (j) use the double angle formula to convert to

$\int 0.0625(1-\cos 2x)^4\,\mathrm{d}x$. expand to $0.0625\int 1-4\cos 2x+6\cos^2 2x-4\cos^3 2x+\cos^4 2x\,\mathrm{d}x$. you have to use double angle formula to expand $\cos^2 2x$. you also have to convert $\cos^3 2x$ to $(1-\sin^2 2x)\cos 2x$.

7. (§7.2) Questions of the form $\int \tan^m x\sec^n x\,\mathrm{d}x$. There are three cases:

(I) If m is odd, keep a $\sec x\tan x$, and convert $\tan^{m-1} x$ to powers of $\sec x$ using $\tan^2 x+1=\sec^2 x$. Use the u substitution $u=\sec x$.

(II) If n is *even*, keep a $\sec^2 x$, and convert $\sec^{n-2} x$ to powers of $\tan x$ using $\tan^2 x+1=\sec^2 x$. Use the u substitution $u=\tan x$.

(III) If m is even and n is odd, protest and complain! This case is difficult. One thing you can do is keep a $\sec x$, and convert $\cos^{1-n} x$ to a denominator in $\sin x$ using $\sin^2 x+\cos^2 x=1$, use $u=\sin x$, then try a partial fraction expansion. Not very pretty.

Questions of this kind: (a) $\int \tan^3 x\sec x\,\mathrm{d}x$. (b) $\int \tan^3 x\sec^3 x\,\mathrm{d}x$. (c) $\int \tan^2 x\sec^4 x\,\mathrm{d}x$. (d) $\int \tan^3 x\sec^6 x\,\mathrm{d}x$. (e) $\int \tan^2 x\,\mathrm{d}x$. (f) $\int \sec^3 x\,\mathrm{d}x$.

Answer: (a) convert to $\int(\sec^2 x-1)\tan x\sec x\,\mathrm{d}x$. use $u=\sec x$. (b) convert to $\int(\sec^4 x-\sec^2 x)\tan x\sec x\,\mathrm{d}x$. use $u=\sec x$. (c) convert to $\int \tan^2 x(\tan^2+1)\sec^2 x\,\mathrm{d}x$. use $u=\tan x$. (d) convert to $\int \tan^3 x(\tan^2+1)^2\sec^2 x\,\mathrm{d}x$. use $u=\tan x$. (e) convert to $\int \sec^2 x-1\,\mathrm{d}x$. (f) not answered yet.

8. (§7.3) 'Hidden' Trig integrals. In these there is usually a component that looks like Pythagora's Theorem, *e.g.*, a $\sqrt{x^2+9}$ or a $\sqrt{25-x^2}$. Draw a triangle with the right sides, make θ one of the angles and express the integral in terms of θ. (a) $\int\sqrt{100-x^2}\,\mathrm{d}x$. (b) $\int\frac{\mathrm{d}x}{\sqrt{9+x^2}}$. (c) $\int\frac{x\,\mathrm{d}x}{\sqrt{9+x^2}}$. (d) $\int\frac{10\,\mathrm{d}x}{x(100+x^2)^{3/2}}$. (e) $\int_0^2 x^3\sqrt{x^2+4}\,\mathrm{d}x$. (f) $\int\frac{\sqrt{1+x^2}\,\mathrm{d}x}{x}$.

Answer: (a) use $\sin\theta=x/10$ to get $\int 100\cos^2\theta\,\mathrm{d}\theta$. (b) use $\tan\theta=x/3$ to get $\int\cos\theta\sec^2\theta\,\mathrm{d}\theta$. This solves as $\ln\left|\sqrt{9+x^2}+x\right|+C$ (c) you can just make the substitution $u=9+x^2$. (d) use $\tan\theta=x/10$ to get $\int\frac{\cos^2\theta\,\mathrm{d}\theta}{100\sin\theta}$ (e) use $\tan\theta=x/2$ to get $32\int_0^{\pi/4}\tan^3\theta\sec^3\theta\,\mathrm{d}\theta$. (f) not answered yet.

9. (§7.3) Sometimes the Hidden Trig integrals are obscured. In these you need to make a substitution to get it to the right form: (a) $\int\frac{x-2}{x^2-2x+5}\,\mathrm{d}x$. (Note that the denominator is irreducible, so a Partial Fraction Expansion will do no good here. Instead, try $u=x-2$.) (b) $\int\frac{\mathrm{d}x}{(x+3)\sqrt{x^2+6x-25}}$. (c) $\int_{-3}^5\frac{\mathrm{d}x}{(55-6x-x^2)^{3/2}}$.

Answer: (a) my hint was a bit off. instead use $u=x-1$. this gives $\int\frac{u-1}{u^2+4}\,\mathrm{d}u$. Break this into two integrals; one is a substitution: $w=u^2+4$; the other is the derivative of $\arctan u/2$. (b) use $u=x+3$ to get $\int\frac{\mathrm{d}u}{u\sqrt{u^2-36}}$. solve with $\sec\theta=u/36$. (c) use $u=x+3$ to get $\int_0^8\frac{\mathrm{d}x}{(64-u^2)^{3/2}}$. then use $\sin\theta=u/8$.

10. (§7.4) Partial Fraction Expansion, *i.e.*, questions of the form $\int\frac{p(x)}{q(x)}\,\mathrm{d}x$ for polynomials p,q. There is a three step strategy:

(I) If $\deg(p)\geq\deg(q)$, then do a 'long division with remainder' to find polynomials s,r such that $p(x)/q(x)=s(x)+r(x)/q(x)$, and $\deg r<\deg q$. Since s can be integrated easily, we need not look at it anymore.

(II) Express $q(x)$ as linear and irreducible quadratic factors. (In this class, the factoring will be either trivial *e.g.*, $q(x) = x^3 - 9x$, or q will be given to you factored.)

(III) Fraction magic! Remember that a linear factor $ax + b$ corresponds to a summand $\frac{K}{ax+b}$, while an irreducible quadratic factor $ax^2 + bx + c$ corresponds to a summand $\frac{Kx+L}{ax^2+bx+c}$. Repeated factors are repeated in the sum.

Examples:

$$\frac{x^2+3x-2}{(x-1)x(x+4)} \Rightarrow \frac{A}{x-1} + \frac{B}{x} + \frac{C}{x+4}$$
$$\frac{2x^2-3}{(x+1)^3(2x-3)} \Rightarrow \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{2x-3}$$
$$\frac{x^4+3x-2}{(x^2+1)(3x+7)(x^2+3x+4)} \Rightarrow \frac{Ax+B}{x^2+1} + \frac{C}{3x+7} + \frac{Dx+E}{x^2+3x+4}$$
$$\frac{x^5-3x^4-2x}{x(x^2+2)^2(x-2)(x+1)^2} \Rightarrow \frac{A}{x} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2} + \frac{F}{x-2} + \frac{G}{x+1} + \frac{H}{(x+1)^2}.$$

Questions of this kind: (a) $\int \frac{x^4+2x^3-3}{x(x-1)(x+3)}\,dx$. (b) $\int \frac{x^3-2x^3-3}{(x-2)(x+1)^2(2x+1)}\,dx$. (c) $\int \frac{2x^2-x+1}{(x^2+3)(x-1)}\,dx$. (d) $\int \frac{-x^2+10x-1}{(x^2+8)(x+1)^2}\,dx$. (e) $\int \frac{-x^3-2x}{(x^2+1)^2(x-1)^2}\,dx$. (f) $\int \frac{x^2}{(x+1)^3}\,dx$. (g) $\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$. (h) $\int_0^1 \frac{x}{x^2+4x+13}\,dx$. (i) $\int \frac{x^2}{x^2+1}\,dx$. (j) $\int \frac{dx}{x^4+x^2}\,dx$. (k) $\int \frac{x^2+3x-2}{(x-1)x(x+4)}\,dx$. (l) $\int \frac{2x^2-3}{(x+1)^3(2x-3)}\,dx$.

Answer: (a) what bother! you have to do a long division to get $\int x + \frac{3x^2-x}{x(x-1)(x+3)}\,dx$. or $\int x + \frac{1/2}{x-1} + \frac{5/2}{x+3}\,dx$. (b) not answered yet. (c) not answered yet. (d) not answered yet. (e) not answered yet. (f) first use

$$\frac{x^2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

multiply by $(x+1)^3$. plug in $x = -1$ to find $C = 1$. then multiply out and compare the left hand and right hand sides:

$$x^2 = A\left(x^2 + 2x + 1\right) + B\,(x+1) + 1$$

compare terms of x^2 and x to find $A = 1, B = -2$. then it's $\int \frac{1}{x+1} + \frac{-2}{(x+1)^2} + \frac{1}{(x+1)^3}\,dx$. make the substitution $u = x + 1$ to get $\int \frac{1}{u} + \frac{-2}{u^2} + \frac{1}{u^3}\,dx$. (g) not answered yet. (h) first use $u = x + 2$ to get $\int_2^3 \frac{u-2}{u^2+9}\,du$. this breaks into a simple substitution and an arctan term. (i) this is just $\int 1 - \frac{1}{x^2+1}\,dx = x - \arctan x + C$. (j) not answered yet. (k) not answered yet. (l) not answered yet.

11. (§7.4) Partial Fraction Expansion may also be obscured. Consider the example

$$\int \frac{dx}{\sqrt[5]{x} + \sqrt[3]{x}}.$$

In this case if you let $u^{15} = x$, you can rewrite this integral as

$$\int \frac{15u^{14}\,du}{u^3 + u^5}.$$

which is a rational function.
Questions of this kind: (a) $\int \frac{e^{3x}-e^{2x}+6e^x}{e^{3x}-e^x}\,\mathrm{d}x$. (b) $\int \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x$.

Answer: (a) use $u = e^x$ to get $\int \frac{u^2-u+6}{u^3-u}\,\mathrm{d}u = \int \frac{u^2-u+6}{u(u+1)(u-1)}\,\mathrm{d}u$. (b) oooops! you can do this by just substituting $u = 1+\cos^2 x$.

12. (§7.5) In reality it may not be immediately obvious how to solve an integral, as they will not be given to you with the relevant section numbers. Thus you should look over the strategies given in §7.5 to approach integrals.
(a) $\int \frac{e^{2x}+4e^x}{e^{2x}+2e^x+5}\,\mathrm{d}x$. (b) $\int \sqrt{3-2x-x^2}\,\mathrm{d}x$. (c) $\int_0^{\pi/4} \tan^5 x \sec^3 x\,\mathrm{d}x$. (d) $\int \arctan x\,\mathrm{d}x$. (e) $\int \frac{x^4}{x^{10}+16}\,\mathrm{d}x$.
(f) $\int \frac{1+e^x}{1-e^x}\,\mathrm{d}x$. (g) $\int x\sin x\,\mathrm{d}x$. (h) $\int (1+\sqrt{x})^3\,\mathrm{d}x$. (i) $\int x^2\sin x\,\mathrm{d}x$. (j) $\int x\sin^2 x\,\mathrm{d}x$. (k) $\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$.
(l) $\int \frac{x}{x^4+4x^2+3}\,\mathrm{d}x$. (m) $\int \ln\left|\frac{x^3-x}{x+1}\right|\,\mathrm{d}x$. (n) $\int_{-1}^{1} x^{-3}\,\mathrm{d}x$. (o) $\int (\ln x)^2\,\mathrm{d}x$. (p) $\int e^x \cos^3 e^x\,\mathrm{d}x$.
(q) $\int \frac{\mathrm{d}x}{x+\sqrt[3]{x}}\,\mathrm{d}x$. (r) $\int \sin\sqrt{x}\,\mathrm{d}x$.

Answer: (a) substitute $u = e^x$ to get a rational function. (b) try $u = x+1$ to get $\int \sqrt{4-u^2}\,\mathrm{d}u$. change to a trig integral. (c) keep a $\sec x\tan x$, *i.e.*, convert to $\int_0^{\pi/4} (\sec^2 x - 1)^2 \sec^2 x \tan x \sec x\,\mathrm{d}x$. then substitute $u = \sec x$. (d) try parts: $u = \arctan x$, $v = x$. (e) use $u = x^5$ to get $\int \frac{1}{5}\frac{\mathrm{d}u}{u^2+16}$. it solves as an arctan. (f) did you try $u = e^x$? this gives $\int \frac{1+u}{u(1-u)}\,\mathrm{d}u$. (g) parts: $u = x$, $\mathrm{d}v = \sin x\,\mathrm{d}x$. (h) multiply it out to get $\int 1 + 3\sqrt{x} + 3x + x^{3/2}\,\mathrm{d}x$. (i) use parts twice. first use $u = x^2$, $\mathrm{d}v = \sin x\,\mathrm{d}x$. (j) use the double angle formula: $\sin^2 x = (1-\cos 2x)/2$. this breaks the integral in two, the latter part of which will require using parts. (k) i think you can substitute $u = 1 - x^2$. (l) first use $u = x^2$ to get $\int \frac{1}{2}\frac{\mathrm{d}u}{u^2+4u+3} = \frac{1}{2}\int \frac{\mathrm{d}u}{(u+3)(u+1)}$. then use partial fraction expansion. ultimately you should get a natural log. (m) not answered yet. (n) this is just a polynomial; integrates easy. (o) parts: $u = (\ln x)^2$, $v = x$. (p) substitute $u = e^x$ to get $\int \cos^3 u\,\mathrm{d}u$. (q) not answered yet. (r) a tricky substitution: $w = \sqrt{x}$ gives you $\int 2w\sin w\,\mathrm{d}w$. then use parts: $u = w$, $\mathrm{d}v = \sin w\,\mathrm{d}w$.

13. (§7.7) Approximate Integration. (a) Approximate the definite integral $\int_0^2 \frac{1}{1+x^2}\,\mathrm{d}x$. using the Trapezoidal Rule, the Midpoint Rule, and Simpson's Rule, with $n = 4$. (Compare to the real answer $\arctan 2 \approx 1.1071$.) (b) Approximate the definite integral $\int_0^1 e^{x^2}\,\mathrm{d}x$. using the Trapezoidal Rule, the Midpoint Rule, and Simpson's Rule, with $n = 6$.

Answer: (a) first $\Delta x = (2-0)/4 = 1/2$. now let $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$, $x_3 = 3/2$, $x_4 = 2$. Then we have

$$\begin{aligned}
T_n &= \frac{1/2}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right], \\
&= \frac{1}{4}\left[1 + 8/5 + 1 + 8/13 + 2/5\right] = \ldots \\
M_n &= (1/2)\left[f(1/4) + f(3/4) + f(5/4) + f(7/4)\right] = \ldots \\
S_n &= \frac{1/2}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right] = \ldots
\end{aligned}$$

(b) not answered yet.