Final Exam Preparation

The second midterm exam is Monday December 6, 11:30am-2:30pm, in the regular class room, WLH 2005. *You must bring a blue book to the exam.* Blue books are available from the bookstore. You also need to bring your student ID card or other form of ID (driver's license, passport, etc.)

The following formulæ will be provided on your exam:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2},$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

Everything else must be committed to memory.

The final exam is comprehensive, covering all the material from this quarter. However, there should be more questions focusing on material we have covered since the second midterm. This preparation sheet covers that material. To review for the remainder of the material, consult the prep sheets from the midterms, and study the midterms.

You should be prepared to answer questions like those that follow. Note that I have not checked many of these problems, some of them may be unsolveable, impractically difficult or generally insane. You are warned.

1. (§7.8) Improper Integrals. Evaluate the following integrals. Tell which are divergent and which are convergent. For convergent integrals, find their value.

$$(a)\int_1^2 \frac{1}{\sqrt{x-1}}\,\mathrm{d}x, \quad (b)\int_{-1}^3 \frac{1}{x^2}\,\mathrm{d}x, \quad (c)\int_1^\infty \frac{x}{1+x^2}\,\mathrm{d}x, \quad (d)\int_{-\infty}^\infty \frac{1}{1+x^2}\,\mathrm{d}x,$$
$$(e)\int_1^\infty \frac{\ln x}{x}\,\mathrm{d}x, \quad (f)\int_8^9 \frac{1}{\sqrt[3]{x-9}}\,\mathrm{d}x, \quad (g)\int_0^\pi \sec x\,\mathrm{d}x, \quad (h)\int_{-1}^1 \frac{e^x}{e^x-1}\,\mathrm{d}x.$$

Answer: (a) Convergent with value 2. (b) Divergent. (c) Divergent. (d) Convergent with value π. (e) Divergent. (f) Convergent with value 3/2. (g) Divergent. (h) Divergent.

2. (§8.1) Arc length. Recall that the arc length of the graph of $y = f(x)$ for $a \le x \le b$ is

$$\int_a^b \sqrt{1+(f'(x))^2}\,\mathrm{d}x.$$

(a) Find the arc length of the graph of $y = x^3/3 + 1/(4x)$ for $1 \le x \le 2$. (b) Find the arc length of the graph of $y = \ln(\sec x)$ for $0 \le x \le \pi/3$. (c) Find the arc length of the graph of $x^2 = 4(y+4)^3$ for $0 \le y \le 2$ with $x > 0$. (d) Find the arc length of the graph of $y = x^2/2$ for $1 \le x \le 2$.

Answer: (a) You should get

$$\int_1^2 x^2 + \frac{1}{4x^2}\,\mathrm{d}x.$$

(b) I get a final answer of $\ln(2+\sqrt{3})$. (c) You should get intermediate integral

$$\frac{1}{9}\int_{33}^{55} \sqrt{u}\,\mathrm{d}u.$$

(d) I think the answer is

$$\int_{1}^{2} \sqrt{1+x^2}\,\mathrm{d}x = \sqrt{5}+\frac{1}{2}\ln\left|2+\sqrt{5}\right| - \frac{\sqrt{2}}{2} - \frac{1}{2}\ln\left|1+\sqrt{2}\right|.$$

3. (§8.2) Surface area of a surface of revolution. Recall that the surface area of the solid obtained by revolving $y = f(x)$ for $a \le x \le b$ about the x-axis is given by

$$2\pi \int_{a}^{b} f(x)\sqrt{1+(f'(x))^2}\,\mathrm{d}x.$$

(a) Find the surface area of the surface obtained by revolving $y = \sqrt{r^2 - x^2}$ for $-r \le x \le r$ about the x axis. (b) Find the surface area of the surface obtained by revolving $y = x$ for $0 \le x \le r$ about the x axis. (c) Find the surface area of the surface obtained by revolving $y = x^3/3$ for $0 \le x \le 2$ about the x axis. (d) Find the surface area of the surface obtained by revolving $x = y^4 + 1/(32y^2)$ for $1 \le y \le 2$ about the y axis. (e) Find the surface area of the surface obtained by revolving $y = \sqrt{x}$ for $1 \le x \le 4$ about the x axis.

Answer: (a) This is just the surface area of a sphere of radius r, or $4\pi r^2$. (b) This is the surface area of a cone of base radius r, and height r, which should have value $\sqrt{2}\pi r^2$. (c) As an intermediate step, you should get

$$\frac{\pi}{6}\int_{1}^{17} \sqrt{u}\,\mathrm{d}u.$$

(d) I get the intermediate integral

$$2\pi \int_{1}^{2} \left(y^4 + \frac{1}{32y^2}\right)\left(4y^3 + \frac{1}{16y^3}\right)\mathrm{d}y = 2\pi \int_{1}^{2} 4y^7 + \frac{3y}{16} + \frac{1}{512y^5}\,\mathrm{d}y.$$

(e) Answer should be the solution of

$$2\pi \int_{1}^{4} \sqrt{x + (1/4)}\,\mathrm{d}x.$$

4. (§8.3) Center of Mass. Recall that the center of mass of the region between the curves $y = f(x)$, $y = 0$, $x = a$, $x = b$ is the point $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{\int_a^b x\, f(x)\, \mathrm{d}x}{\int_a^b f(x)\, \mathrm{d}x}, \qquad \overline{y} = \frac{\frac{1}{2}\int_a^b (f(x))^2\, \mathrm{d}x}{\int_a^b f(x)\, \mathrm{d}x}.$$

Find the center of mass of the following regions: (a) the region bounded by $y = 4 - x^2$ and the x axis; (b) the region bounded by $y = e^x$, the x axis, $x = 0$, and $x = 1$;

Answer: (a) The center of mass, $(\bar{x}, \bar{y})$, is $(0, 8/5)$. You can find $\bar{x}$ quickly by symmetry, as this region is symmetric with respect to the y axis. To find $\bar{y}$ you have to evaluate the integrals. (b) I get

$$\left(\frac{1}{e-1}, \frac{e+1}{4}\right).$$

5. (§8.5) Probability. A random variable, X has probability distribution function f if the probability that X takes value in $[a, b]$ is

$$P\,(a \leq X \leq b) = \int_a^b f(x)\, \mathrm{d}x.$$

The function $f(x)$ should have these properties: $f(x) \geq 0$ for all x, and

$$\int_{-\infty}^{\infty} f(x)\, \mathrm{d}x = 1.$$

Verify that the following functions are probability density functions:

(a) for any $\alpha < \beta$,

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha \leq x \leq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

(b) $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$.

(c) for any $j > 1$,

$$f(x) = \begin{cases} (j-1)x^{-j} & \text{if } 1 \leq x, \\ 0 & \text{otherwise,} \end{cases}$$

6. (§8.5) Probability. The *mean* (or expected value) of a random variable, X, with probability distribution function f, is

$$\int_{-\infty}^{\infty} x f(x)\, \mathrm{d}x.$$

Find the mean of each of the following random variables:

(a) X with probability distribution function

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \alpha \leq x \leq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

(b) X with probability distribution function

$$f(x) = \begin{cases} (j-1)x^{-j} & \text{if } 1 \leq x, \\ 0 & \text{otherwise,} \end{cases}$$

for any $j > 1$.

Answer: (a) $(\beta + \alpha)/2$.

(b) If $1 < j \leq 2$, the mean calculation gives a divergent improper integral. We did not talk about this possibility in class, and you need not worry about it. If $j > 2$, the mean is $(j-1)/(j-2)$.

7. (§10.5) Conic Sections. Identify the following equations as being those of an ellipse, parabola, or hyperbola. Then sketch the graph of each of them.

(a) $4x^2 = -y$,
(b) $(x/4)^2 + (y/9)^2 = 1$,
(c) $x = y^2 - 4y + 4$,
(d) $-(x/9)^2 + (y/1)^2 = 1$,
(e) $x^2 + y^2 = 100$,
(f) $-x^2 + y^2 = 36$,

Answer: (a) a parabola with axis along the y-axis; (b) an ellipse with foci on the y-axis; (c) a parabola with axis along $y = 2$; (d) a hyperbola with foci on the y-axis; (e) a circle, a special case of the ellipse; (f) a hyperbola with foci along the y-axis.

8. (§9.2) Direction Fields. Consider the four direction fields plotted in Figure 1. Match each field to one of the following differential equations:

(I) $y' = y\sin(2x)$,
(II) $y' = y^2 - x^2$,
(III) $y' = y\,(1 - y/2)$,
(IV) $y' = 1 - xy$.

Answer: (I) Figure 1(b),
(II) Figure 1(c),
(III) Figure 1(a),
(IV) Figure 1(d).

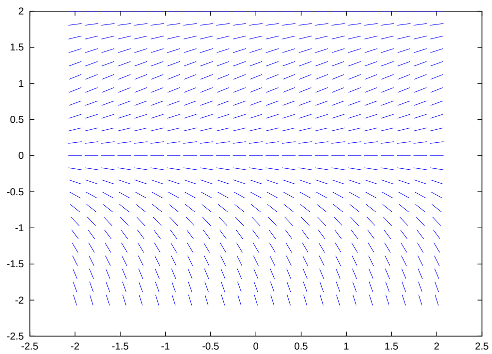
9. (§9.2) Direction Fields. On each of the direction fields plotted in Figure 1, sketch the graph of the function satisfying the differential equation and which goes through the point $(0, 0.5)$.

10. (§9.3) Separable Differential Equations.
Solve the following differential equations by separation.

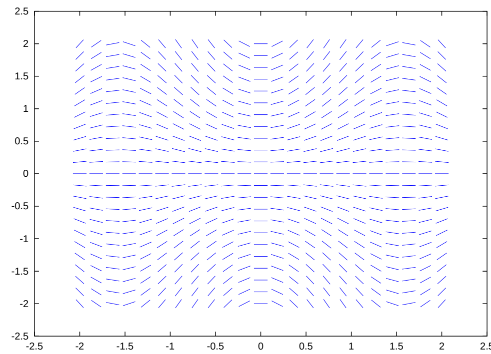
(a) $y' = yx$,
(b) $y' = x^2/y^2$,

(c) $y' = y^2 \sin x$,
(d) $y' = \sqrt{xy}$,

Answer: (a) $y = Ce^{x^2/2}$,
(b) $y = \sqrt[3]{x^3 + C}$,
(c) $y = (\cos x + C)^{-1}$,
(d) $y = \left(x^{3/2}/3 + C\right)^2$.



(a)



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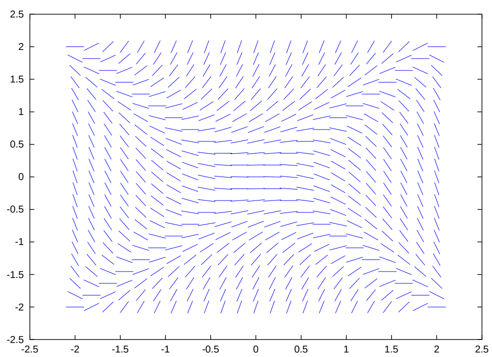

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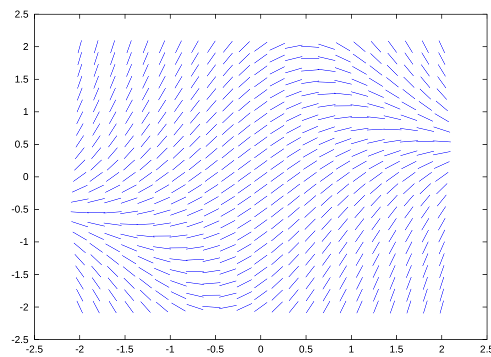

(d)

Figure 1: Four direction fields.