Exam 1		**S 2004 M172 : Numerical PDE**
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Name ____________________________ Student #________________

Instructions: Read all instructions carefully. Write your name and student number above. Clearly indicate your answers & show all your work on your answer sheet. For many problems partial credit is available. 6 Problems worth 100 Points.

Grading Notes: For those questions with multiple parts, please circle or box your answers so that I do not have to search for them.

Hints:

$$\cos\theta - 1 = -2\sin^2(\theta/2)$$

Problems. Show all work on your answer sheets. Partial credit is available.

P1 (**10 pnts**) Write down *two* of the following finite difference schemes:

- Explicit scheme for Heat Equation.
- Implicit scheme for Heat Equation.
- Crank Nicolson scheme for Heat Equation.
- FTFX for Transport Equation.
- Upwind Scheme for Transport Equation.
- Lax Wendroff Scheme for Transport Equation.

You will get no credit for writing down more than two schemes. Your answer need only consist of which schemes you are writing and a single equation for each scheme.

P2 (**10 pnts**) Complete the statement of Taylor's Theorem:
Suppose $f(x)$ has   continuous derivatives on $[a,b]$. Then for $x, x+h$ in $[a,b]$

$$f(x+h) = \left(\sum_{j=}^{n} \frac{1}{(\ )} h^{(\ )} f^{(\ )}(\ )\right) + \frac{1}{(\ )} h^{(\ )} f^{(\ )}(\ ),$$

where ξ is some number between x and $x+h$.

P3 (**15 pnts**) (a) Use Taylor's Theorem to derive a $\mathcal{O}\left(h^2\right)$ difference approximation for $f'(x)$.

(b) Suppose $f(x)$ has the property that $|f'''(x)| \le 1$. Can you give a specific bound for the error of the approximation you made above, *i.e.*, find the number K such that the error of your approximation is no more than Kh^2.

Exam continues on reverse of page.

P4 (**25 pnts**) Recall the transport equation, where we assume a is a constant:

$$\begin{cases} u_t + au_x = 0 & x \in \mathbb{R},\, t > 0 \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases}$$

Consider the funny scheme to solve the advection equation:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_{j+2}^n - U_{j+1}^n}{\Delta x} = 0$$

(a) What is the Domain of Dependance for the PDE at point (x_i, t_j)? (*Hint:* it is a singleton.)
(b) What is the Domain of Dependance for this scheme at point (x_i, t_j)?
(c) State the CFL condition. Under what conditions on $\nu = a\Delta t/\Delta x$ does this scheme satisfy the CFL condition?
(d) Does the CFL condition imply stability? If not, perform an analysis to find conditions on ν that imply stability. (*Hint:* If you plan on using a Fourier Analysis, you may assume that β is chosen such that $e^{i\beta\Delta x}$ is purely real.)

P5 (**30 pnts**) Perform the following Fourier Analysis:

(a) Define the symbol "δ_x^2" as follows: $\delta_x^2\, U_j^n =_{\mathrm{df}} U_{j+1}^n - 2U_j^n + U_{j-1}^n$. Assuming $U_j^n = \lambda^n e^{i\beta j\Delta x}$, simplify $\delta_x^2\, U_j^n$ in terms of $\lambda, \beta, j, n, \Delta x$.
(b) Assuming $U_j^n = \lambda^n e^{i\beta j\Delta x}$, reduce the following finite difference scheme to a quadratic equation in λ:

$$\frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} = \frac{1}{4(\Delta x)^2}\left(\delta_x^2\, U_j^{n+1} + 2\delta_x^2\, U_j^n + \delta_x^2\, U_j^{n-1}\right)$$

(*Hint:* Your answer should look like $(k-1)\lambda^2 + 2k\lambda + (k+1) = 0$, for some $k < 0$.)
(c) This equation has two roots; are they both real or imaginary? Based on this answer, do you think the PDE which is approximated by this scheme is hyperbolic (has moving component) or parabolic (no moving component)?
(d) Prove that this scheme is unconditionally stable, *i.e.*, that $|\lambda| \leq 1$ for all $\Delta t, \Delta x$.

P6 (**10 pnts**) State some substantive question which you thought might appear on this exam, but did not. Answer this question (correctly).