Exam 1		**S 2004 M172 : Numerical PDE**
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Name ______________________ Student #__________________

Instructions: Read all instructions carefully. Write your name and student number above. Clearly indicate your answers & show all your work on your answer sheet. For many problems partial credit is available. 6 Problems worth 100 Points.

Grading Notes: For those questions with multiple parts, please circle or box your answers so that I do not have to search for them.

Hints:

$$\cos\theta - 1 = -2\sin^2(\theta/2)$$

Problems. Show all work on your answer sheets. Partial credit is available.

P1 (**10 pnts**) Write down *two* of the following finite difference schemes:

- Explicit scheme for Heat Equation.
  answer:
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}$$
- Implicit scheme for Heat Equation.
  answer:
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Delta x^2}$$
- Crank Nicolson scheme for Heat Equation.
  answer:
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{1}{2}\left(\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}\right) + \frac{1}{2}\left(\frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Delta x^2}\right)$$
- FTFX for Transport Equation.
  answer:
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = -a\frac{U_{j+1}^{n+1} - U_j^{n+1}}{2\Delta x}$$
- Upwind Scheme for Transport Equation.
  answer:
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{-a}{2\Delta x}\begin{cases} U_{j+1}^{n+1} - U_j^{n+1} & \text{if } a < 0, \\ U_j^{n+1} - U_{j-1}^{n+1} & \text{if } a > 0. \end{cases}$$
- Lax Wendroff Scheme for Transport Equation.
  answer:
$$U_j^{n+1} = \frac{1}{2}\nu(1+\nu)U_{j-1}^n + (1-\nu^2)U_j^n + -\frac{1}{2}\nu(1-\nu)U_{j+1}^n, \quad \text{where } \nu = a\Delta t/\Delta x.$$

You will get no credit for writing down more than two schemes. Your answer need only consist of which schemes you are writing and a single equation for each scheme.

P2 (**10 pnts**) Complete the statement of Taylor's Theorem:

Suppose $f(x)$ has   continuous derivatives on $[a,b]$. Then for $x, x+h$ in $[a,b]$

$$f(x+h) = \left(\sum_{j=}^{n} \frac{1}{(\ )} h^{(\ )} f^{(\ )}(\ ) \right) + \frac{1}{(\ )} h^{(\ )} f^{(\ )}(\ ),$$

where ξ is some number between x and $x+h$.

answer: Suppose $f(x)$ has $n+1$ continuous derivatives on $[a,b]$. Then for $x, x+h$ in $[a,b]$

$$f(x+h) = \left(\sum_{j=0}^{n} \frac{1}{j!} h^j f^{(j)}(x) \right) + \frac{1}{(n+1)!} h^{n+1} f^{(n+1)}(\xi),$$

where ξ is some number between x and $x+h$.

P3 (**15 pnts**) (a) Use Taylor's Theorem to derive a $\mathcal{O}\left(h^2\right)$ difference approximation for $f'(x)$.

answer:

$$\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^3 f'''(\xi_+) \\
f(x-h) &= f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) - \frac{1}{6}h^3 f'''(\xi_-) \\
f(x+h) - f(x-h) &= 2hf'(x) + \frac{1}{6}h^3 \left(f'''(\xi_+) + f'''(\xi_-) \right) \\
\frac{f(x+h) - f(x-h)}{2h} &= f'(x) + \frac{1}{12}h^2 \left(f'''(\xi_+) + f'''(\xi_-) \right)
\end{aligned}$$

(b) Suppose $f(x)$ has the property that $|f'''(x)| \leq 1$. Can you give a specific bound for the error of the approximation you made above, *i.e.*, find the number K such that the error of your approximation is no more than Kh^2.

answer: Bounding $|f'''(\xi_+)|$ and $|f'''(\xi_+)|$ by 1 gives $K = \frac{1}{6}$.

P4 (**25 pnts**) Recall the transport equation, where we assume a is a constant:

$$\begin{cases} u_t + au_x = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases}$$

Consider the funny scheme to solve the advection equation:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a \frac{U_{j+2}^n - U_{j+1}^n}{\Delta x} = 0$$

(a) What is the Domain of Dependance for the PDE at point (x_i, t_j)? (*Hint:* it is a singleton.)

answer: The domain is the singleton $\{x_i - at_j\}$.

(b) What is the Domain of Dependance for this scheme at point (x_i, t_j)?

answer: See Figure 1. The Domain of Dependence is $[x_i, x_{i+2j}]$.

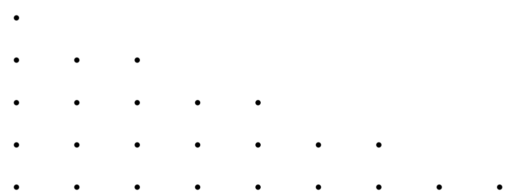


Figure 1: The stencil of the funny scheme extended. The Domain of Dependence of the scheme at (x_i, t_j) is $[x_i, x_{i+2j}]$.

(c) State the CFL condition. Under what conditions on $\nu = a\Delta t/\Delta x$ does this scheme satisfy the CFL condition?
answer: The CFL condition states that "for a convergent finite difference scheme the Domain of Dependance of the PDE must lie inside that of the difference scheme."
This occurs when

$$\begin{array}{rrcl} & x_i \leq & x_i - at_j & \leq x_{i+2j} \\ \Leftrightarrow & i\Delta x \leq & i\Delta x - aj\Delta t & \leq (i+2j)\,\Delta x \\ \Leftrightarrow & 0 \leq & -aj\Delta t & \leq 2j\Delta x \\ \Leftrightarrow & 0 \leq & -a\Delta t & \leq 2\Delta x \\ \Leftrightarrow & 0 \leq & -\nu & \leq 2 \\ \Leftrightarrow & -2 \leq & \nu & \leq 0 \end{array}$$

(d) Does the CFL condition imply stability? If not, perform an analysis to find conditions on ν that imply stability. (*Hint:* If you plan on using a Fourier Analysis, you may assume that β is chosen such that $e^{i\beta\Delta x}$ is purely real.)
answer: The CFL condition does *not* imply stability. We perform a Fourier Analysis. Suppose $U_j^n = \lambda^n e^{i\beta j\Delta x}$. Then our scheme becomes

$$\begin{aligned} \frac{U_j^{n+1} - U_j^n}{\Delta t} &= -a\frac{U_{j+2}^n - U_{j+1}^n}{\Delta x} \\ \lambda^{n+1}e^{i\beta j\Delta x} - \lambda^n e^{i\beta j\Delta x} &= -\nu\left(\lambda^n e^{i\beta(j+2)\Delta x} - \lambda^n e^{i\beta(j+1)\Delta x}\right) \\ \lambda^n e^{i\beta j\Delta x}(\lambda - 1) &= -\nu\lambda^n e^{i\beta j\Delta x}\left(e^{i\beta 2\Delta x} - e^{i\beta\Delta x}\right) \\ (\lambda - 1) &= -\nu e^{i\beta\Delta x}\left(e^{i\beta\Delta x} - 1\right) \\ \lambda &= 1 - \nu\cos(\beta\Delta x)\left[\cos(\beta\Delta x) - 1\right], \end{aligned}$$

In the last line we used the assumption that $e^{i\beta\Delta x}$ was purely real, *i.e.*, $\cos(\beta\Delta x)$.
For $\beta\Delta x$ near pi, we have $\cos(\beta\Delta x)$ positive and nearly 2. But since we need ν to be negative to satisfy CFL, this means that $\lambda > 1$, so we do not have stability. Thus there is no choice of ν that satisfies the CFL condition and gives stability, which makes this scheme useless.

P5 (**30 pnts**) Perform the following Fourier Analysis:

(a) Define the symbol "δ_x^2" as follows: $\delta_x^2 U_j^n =_{\mathrm{df}} U_{j+1}^n - 2U_j^n + U_{j-1}^n$. Assuming $U_j^n = \lambda^n e^{i\beta j\Delta x}$, simplify $\delta_x^2 U_j^n$ in terms of $\lambda, \beta, j, n, \Delta x$.

answer: What I was looking for was

$$\begin{aligned}
\delta_x^2 U_j^n &= U_{j+1}^n - 2U_j^n + U_{j-1}^n, \\
&= \lambda^n e^{i\beta(j+1)\Delta x} - 2\lambda^n e^{i\beta j\Delta x} + \lambda^n e^{i\beta(j-1)\Delta x} \\
&= \lambda^n e^{i\beta j\Delta x}\left(e^{i\beta\Delta x} - 2 + e^{i\beta\Delta x}\right) \\
&= \lambda^n e^{i\beta j\Delta x}\left(\cos(\beta\Delta x) + i\sin(\beta\Delta x) - 2 + \cos(\beta\Delta x) - i\sin(\beta\Delta x)\right) \\
&= \lambda^n e^{i\beta j\Delta x}\left(2\cos(\beta\Delta x) - 2\right) \\
&= \lambda^n e^{i\beta j\Delta x}\left(-4\sin^2(\beta\Delta x/2)\right) \\
&= -4\lambda^n e^{i\beta j\Delta x}\sin^2(\beta\Delta x/2)
\end{aligned}$$

(b) Assuming $U_j^n = \lambda^n e^{i\beta j\Delta x}$, reduce the following finite difference scheme to a quadratic equation in λ:

$$\frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} = \frac{1}{4(\Delta x)^2}\left(\delta_x^2 U_j^{n+1} + 2\delta_x^2 U_j^n + \delta_x^2 U_j^{n-1}\right)$$

(*Hint:* Your answer should look like $(k-1)\lambda^2 + 2k\lambda + (k+1) = 0$, for some $k < 0$.)

answer: What I was looking for was

$$\begin{aligned}
\frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} &= \frac{1}{4(\Delta x)^2}\left(\delta_x^2 U_j^{n+1} + 2\delta_x^2 U_j^n + \delta_x^2 U_j^{n-1}\right) \\
\lambda^{n+1}e^{i\beta j\Delta x} - \lambda^{n-1}e^{i\beta j\Delta x} &= \frac{\Delta t}{2(\Delta x)^2}\left(-4\lambda^n e^{i\beta j\Delta x}\sin^2(\beta\Delta x/2)\right)\left(\lambda + 2 + \lambda^{-1}\right) \\
\lambda - \lambda^{-1} &= \frac{\Delta t}{(\Delta x)^2}\left(-2\sin^2(\beta\Delta x/2)\right)\left(\lambda + 2 + \lambda^{-1}\right) \\
\lambda^2 - 1 &= \left[\frac{-2\Delta t}{(\Delta x)^2}\sin^2(\beta\Delta x/2)\right]\left(\lambda^2 + 2\lambda + 1\right) \\
\lambda^2 - 1 &= k\left(\lambda^2 + 2\lambda + 1\right) \\
0 &= (k-1)\lambda^2 + 2k\lambda + (k+1),
\end{aligned}$$

where

$$k = \left[\frac{-2\Delta t}{(\Delta x)^2}\sin^2(\beta\Delta x/2)\right].$$

(c) This equation has two roots; are they both real or imaginary? Based on this answer, do you think the PDE which is approximated by this scheme is hyperbolic (has moving component) or parabolic (no moving component)?

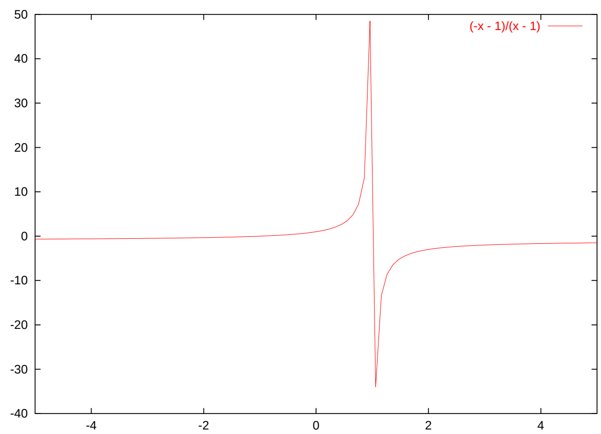
answer: You really did not have to answer the last part correctly to get this one. The roots are

$$\lambda = \frac{-2k \pm \sqrt{4k^2 - 4(k-1)(k+1)}}{2(k-1)} = \frac{-k \pm 1}{k-1}.$$

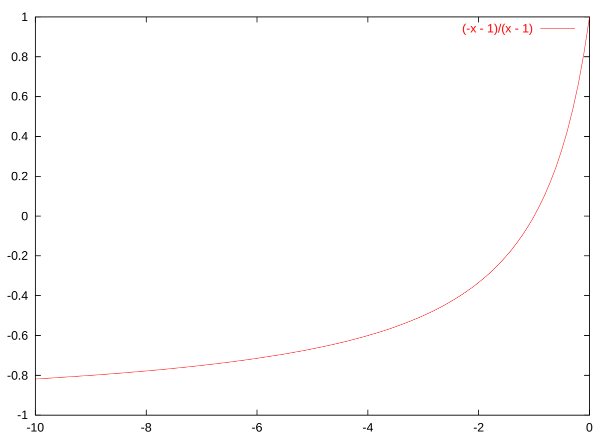
Thus the roots are real. This means that the scheme is *parabolic*. Note this is different than our classification of PDEs based on some form $Au_{xx} + Bu_{xy} + Cu_{yy} + \cdots = 0$. Because λ is real, there is no motion components, just damping. In fact, you can see this scheme models the heat equation $u_t = u_{xx}$.

(d) Prove that this scheme is unconditionally stable, *i.e.*, that $|\lambda| \leq 1$ for all $\Delta t, \Delta x$.

answer: One of the roots is $\lambda_1 = \frac{-k+1}{k-1} = -1$. The other is $\lambda_2 = \frac{-k-1}{k-1}$, where k is negative. Recall the days of precalculus when you had to graph quotient functions like this. The graph of $\frac{-x-1}{x-1}$ if shown in Figure 2. For $x \leq 0$ it takes values only on $(-1, 1]$. Thus $|\lambda_2| \leq 1$. Thus the scheme is unconditionally stable, as the amplification factor $|\lambda| \leq 1$.



(a) wide interval



(b) detail

Figure 2: The graph of $\frac{-x-1}{x-1}$ is shown.

P6 (**10 pnts**) State some substantive question which you thought might appear on this exam, but did not. Answer this question (correctly).