Exam 1 Prep	**S 2004 M172 : Numerical PDE**
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The following look like good exam type questions and material.

1. What does it mean for a problem to be "well posed" for a given PDE? State a well-posed problem, and one which is not well posed.
2. What does it mean for a curve $(\xi(t), t)$ to be a characteristic of a PDE?
3. Give the closed form solution to the transport equation:

$$\begin{cases} u_t + a u_x = 0 & x \in \mathbb{R},\, t > 0 \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases}$$

   What are the characteristics of this equation?
4. Give the closed form solution to the wave equation:

$$\begin{cases} u_{xx} - \frac{1}{c^2} u_{tt} = 0 & x \in \mathbb{R},\, t > 0 \\ u(x,0) = \phi(x) & x \in \mathbb{R}, \\ u_t(x,0) = \psi(x) & x \in \mathbb{R}, \end{cases}$$

   What are the characteristics of this equation?
5. Give the classification scheme for second order quasilinear PDEs of the type:

$$A u_{xx} + B u_{xy} + C u_{yy} + D U_x + E U_y + F U + G = 0$$

   Classify the following PDE's as hyperbolic, parabolic, or elliptic, renaming t to be y if necessary:
   (a) $u_{xx} - u_{yy} = 0.$
   (b) $u_{xx} + u_{yy} = 0.$
   (c) $u_t - u_{xx} = f.$
   (d) $u_t - u_x = f.$
   (e) $u_{xx} - \frac{1}{c^2} u_{tt} = 0.$
   (f) $u_{xx} + u u_{xy} + u_y y = f.$
6. State Taylor's Theorem for $f(x+h)$. Use this theorem to get a $\mathcal{O}(h)$ approximation of $f'(x)$. Use it to get a $\mathcal{O}\left(h^2\right)$ approximation to $f''(x)$.
7. Use Taylor's Theorem to find the form of the following differences:

$$\begin{aligned} \frac{f(x+h) - f(x-h)}{2h} &= f'(x) + a_1 h^2 f'''(x) + \mathcal{O}\left(h^4\right) \\ \frac{f(x+2h) - f(x-2h)}{4h} &= f'(x) + a_2 h^2 f'''(x) + \mathcal{O}\left(h^4\right) \end{aligned}$$

   Combine these differences to get a $\mathcal{O}\left(h^4\right)$ difference approximation, *i.e.*, find β_1, β_2 such that

$$\beta_1 \frac{f(x+h) - f(x-h)}{2h} + \beta_2 \frac{f(x+2h) - f(x-2h)}{4h} = f'(x) + \mathcal{O}\left(h^4\right)$$

8. What does it mean for $\boldsymbol{v}, \lambda$ to be eigenvector and eigenvalue of matrix A? What are the eigenvalues and -vectors of the identity matrix I? Suppose $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of matrix A. What are the eigenvalues of $\mathsf{B} = \alpha \mathsf{I} + \beta \mathsf{A}$?

9. Let $\{\boldsymbol{v_i}\}_{i=1}^n$ be the eigenvectors of matrix A. Suppose these vectors span $\mathbb{R}^n$. Suppose the eigenvalues of A are all nonnegative and real. What can you say about
$$\boldsymbol{x}^\top \mathsf{A} \boldsymbol{x}$$
for arbitrary vector $\boldsymbol{x}$?
10. Suppose the eigenvalues of a matrix are all positive and real. Can the matrix be singular? Suppose A is a singular matrix; can you name one of its eigenvalues?
11. What are the different errors that might occur in a finite difference scheme? Do you think they can be minimized independently? How do you define consistency? Stability?
12. Give the explicit scheme for the solution of the Parabolic nonhomogeneous heat equation with boundary values:
$$\begin{cases} u_t - u_{xx} = f(x,t) & x \in [0,1], \, t \in [0,T], \\ u(x,0) = u_0(x) & x \in [0,1], \\ u(0,t) = a(t) & t \in (0,T], \\ u(1,t) = b(t) & t \in (0,T] \end{cases}$$
Write the scheme as
$$\boldsymbol{U^{j+1}} = \mathsf{A}\boldsymbol{U^j} + \boldsymbol{F}^j$$
for some matrix A, and some starting vector $\boldsymbol{U^0}$. What are the eigenvectors,-values of this matrix?
13. Perform a stability analysis for this explicit scheme. Suppose that some error occurs at the first step, and your program starts with $\boldsymbol{\tilde{U}^0}$, then computes $\boldsymbol{\tilde{U}^j}$ flawlessly according to this scheme. Let $\boldsymbol{Z^j} = \boldsymbol{U^j} - \boldsymbol{\tilde{U}^j}$. Give a finite difference scheme for $\boldsymbol{Z^j}$. Find the necessary and sufficient condition for stability, *i.e.*, find conditions on $\Delta t, \Delta x$ such that $\left\|\boldsymbol{Z^j}\right\| \to 0$, as $j \to \infty$.
You should be prepared to do this three ways:
    (a) Directly using a "sup-norm" argument: let $z^j = \max_i \left|Z_i^j\right|$. Find conditions such that $z^{j+1} \leq z^j$.
    (b) Using an eigenvalue argument: find conditions such that the eigenvalues of A are sufficiently small.
    (c) By Fourier Analysis: pretend that U_j^n takes the form $\lambda^n e^{i\beta j \Delta x}$, and find conditions such that $|\lambda|$ is sufficiently small.
14. Give the implicit scheme and θ-scheme for the heat equation. Perform stability analyses for these by the Fourier method.
15. Set up the method of lines solution to the *Burgers' Equation*:
$$\begin{cases} -uu_x + \nu u_{xx} = u_t & x \in [0,1], \, t \in [0,T], \\ u(x,0) = u_0(x) & x \in [0,1], \\ u(0,t) = a(t) & t \in (0,T], \\ u(1,t) = b(t) & t \in (0,T] \end{cases}$$
Divide the unit interval into M pieces, and set up a system of ODEs for the functions $\{U_l(t)\}_{l=0}^M$, which are supposed to approximate $u(x_l, t) = u(l/M, t)$, with u the solution to the PDE. This is a difficult system, do not attempt to solve it.

16. State the CFL condition. Is this condition sufficient for stability and convergence? Consider the transport equation:

$$\begin{cases} u_t + au_x = 0 & x \in \mathbb{R},\, t > 0 \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases}$$

What is the Domain of Dependance of this PDE, at a given point (x_i, t_j)?
Give the FTFX scheme for this problem. What is the Domain of Dependance for this scheme? Under what conditions are the CFL condition satisfied for this scheme? By Fourier analysis, under what conditions is this scheme stable?
17. Suppose that the FTFX scheme is applied to the transport equation, and the mesh sizes are chosen such that $a\Delta t = -\Delta x$. Argue that the scheme is exact in this case, *i.e.*, that there is no error.
18. Let $\{\boldsymbol{v_i}\}_{i=1}^n$, $\{\lambda_i\}_{i=1}^n$ be the eigenvectors, -values of matrix A. Suppose $\lambda_1 > 1$, and $|\lambda_i| < 1$ for $i = 2, 3, \ldots, n$. Let $\boldsymbol{U^0}$ be some random vector. Make an argument that

$$\boldsymbol{U^n} = \mathsf{A}^n \boldsymbol{U^0}$$

tends towards $\lambda_1^n \boldsymbol{v_1}$.
Now suppose you are considering a "conservative" finite difference scheme for the transport equation; this scheme is explicit, and counteracts damping by unitizing at each iterate. That is, $\boldsymbol{U^0}$ is a unit vector that corresponds to the initial value data of the problem, and at each time step you make the approximation:

$$\boldsymbol{V^{j+1}} = \mathsf{A}\boldsymbol{U^j}, \qquad \boldsymbol{U^{j+1}} = \frac{\boldsymbol{V^{j+1}}}{\left\|\boldsymbol{V^{j+1}}\right\|}$$

Make some guesses about what happens to $\boldsymbol{U^j}$ as $j \to \infty$.
19. The *total variation* of a bi-infinite sequence $\{U_i\}_{i=-\infty}^{\infty}$ is the sum

$$\sum_i |U_i - U_{i-1}|$$

A differencing scheme is said to be *total variation diminishing* if

$$\sum_i \left|U_i^{j+1} - U_{i-1}^{j+1}\right| \leq \sum_i \left|U_i^j - U_{i-1}^j\right|$$

Show that the FTFX scheme for the advection equation has this property when it is stable, *i.e.*, when $-\Delta x \leq a\Delta t \leq 0$. Show that the Lax Wendroff scheme does not have this property for any Δt.