

Name _____ Student # _____

Instructions: Read all instructions carefully. Write your name and student number above. Clearly indicate your answers & show all your work on your answer sheet. For many problems partial credit is available. 5 Problems worth 100 Points.

Grading Notes: For those questions with multiple parts, please circle or box your answers so that I do not have to search for them.

Hints:

$$\cos \theta - 1 = -2 \sin^2 (\theta/2)$$

Problems. Show all work on your answer sheets. Partial credit is available.

P1 (10 pnts) State what it means for a problem to be “well-posed” for a PDE. Give an example of a well-posed problem. Your answer should consist of both a PDE and the problem.

P2 (30 pnts) Perform the following Fourier Analysis:

- (a) Define the symbol “ δ_k^2 ” as follows: $\delta_k^2 U_j^n =_{\text{df}} U_{j+k}^n - 2U_j^n + U_{j-k}^n$. Assuming $U_j^n = \lambda^n e^{i\beta j \Delta x}$, simplify $\delta_k^2 U_j^n$ in terms of $\lambda, \beta, j, n, k, \Delta x$.
- (b) Define the symbol “ δ^2 ” as follows: $\delta^2 U_j^n =_{\text{df}} 4\delta_1^2 U_j^n - \delta_2^2 U_j^n$. Assuming $U_j^n = \lambda^n e^{i\beta j \Delta x}$, simplify $\delta^2 U_j^n$ in terms of $\lambda, \beta, j, n, \Delta x$. (We do this because $(\delta^2 U_j^n)/(3\Delta x^2)$ is an $\mathcal{O}(\Delta x^4)$ approximation to u_{xx} at (x_j, t_n) .)
- (c) Assuming $U_j^n = \lambda^n e^{i\beta j \Delta x}$, reduce the following finite difference scheme to an equation involving λ :

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{1}{3(\Delta x)^2} \delta^2 U_j^n$$

- (d) Find conditions on $\Delta t, \Delta x$ such that this scheme is stable, *i.e.*, such that $|\lambda| \leq 1$. You may wish to use the fact that

$$0 \leq 4 \sin^2 \frac{\theta}{2} + \sin^2 \theta \leq 4.$$

Exam continues on reverse of page.

P3 (10 pnts) Complete the statement of Taylor's Theorem:

Suppose $f(x)$ has n continuous derivatives on $[a, b]$. Then for $x, x + h$ in $[a, b]$

$$f(x+h) = \left(\sum_{j=0}^n \frac{1}{j!} h^j f^{(j)}(\xi) \right) + \frac{1}{(n+1)!} h^{n+1} f^{(n+1)}(\xi),$$

where ξ is some number between x and $x+h$.

- P4 (15 pnts) (a) Use Taylor's Theorem to derive a $\mathcal{O}(h^2)$ difference approximation for $f''(x)$.
 (b) Suppose $f(x)$ has the property that $|f^{(4)}(x)| \leq 12$. Can you give a specific bound for the error of the approximation you made above, i.e., find the number K such that the error of your approximation is no more than Kh^2 .

P5 (35 pnts) Consider the PDE:

$$\begin{cases} u_t + 2txu_x = 0 & x \in \mathbb{R}, t > 0, \\ u(x, 0) = u_0(x) & x \in \mathbb{R}. \end{cases}$$

We wish to solve this PDE for $(x, t) \in \mathcal{D} =_{\text{df}} [0, 2] \times [0, 2]$. We will approximate the solution with the FTBX scheme:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a(x_j, t_n) \frac{U_j^n - U_{j-1}^n}{\Delta x} = 0,$$

with $a(x, t) = 2tx$.

- (a) Show that $u(x, t) = u_0(xe^{-t^2})$ is a solution to this PDE.
 (b) We define a curve $(x, t) = (\xi(t), t)$ for $t \in [0, 2]$, to be a characteristic curve of $u_t + a(x, t)u_x = 0$ if
 i. $a(\xi(t), t) = \xi'(t)$ for $t \in [0, 2]$.
 ii. $u(\xi(t), t)$ is constant for $t \in [0, 2]$.
 Show that $\xi(t) = e^{c+t^2}$ defines a characteristic curve of this PDE for any c .
 (c) What is the Domain of Dependence for the PDE at point (x_j, t_n) ? (*Hint*: it is a singleton.)
 (d) What is the Domain of Dependence, at point (x_j, t_n) for this discretization scheme?
 (e) State the CFL condition. Does the CFL condition imply stability?
 (f) Find a reasonably small K such that you can prove

$$0 \leq a(x, t) \leq K$$

for $(x, t) \in \mathcal{D}$. (This should be simple.)

- (g) Show that the CFL condition is satisfied for $(x_j, t_n) \in \mathcal{D}$ when $K\Delta t \leq \Delta x$. (*Hint*: this is tricky. One way to do it is to prove that the Domain of Dependence for the PDE at (x_j, t_n) is between $x_j - Kt_n$ and x_j , and go from there. To prove this claim you might have to use some elementary calculus, taking the derivatives of two functions with respect to t . There may be other ways of proving this, however.)