Exam 2	**S 2004 M172 : Numerical PDE**
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Name ______________________ Student #______________

Instructions: Read all instructions carefully. Write your name and student number above. Clearly indicate your answers & show all your work on your answer sheet. For many problems partial credit is available. 6 Problems worth 100 Points.

Grading Notes: For those questions with multiple parts, please circle or box your answers so that I do not have to search for them.

Problems. Show all work on your answer sheets. Partial credit is available.

P1 (**10 pnts**) Give a discretization of the one dimensional elliptic "P"DE

$$\begin{cases} u_{xx} = f & x \in (0,1)\,, \\ u(0) = 0 \\ u(1) = 1 \end{cases}$$

Your discretization should be of the form

$$\mathsf{A}\boldsymbol{U} = \boldsymbol{b}$$

By Taylor's theorem, what is the size of the discretization error?

P2 (**10 pnts**) Give *either* the Ritz (weak) formulation or the Galerkin (weak) formulation of the PDE

$$\begin{cases} -\nabla^2 u = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

You do *not* need to give both, or derive either, or show they are equivalent.

P3 (**20 pnts**) Suppose you are trying to solve the linear system

$$\mathsf{A}\boldsymbol{x} = \boldsymbol{b},$$

where A is an $n \times n$ symmetric, positive definite matrix.

(a) Define what "positive definite" means.

(b) Give pseudocode for one of the following iterative solvers:

i. Jacobi,
ii. Gauss Seidel,
iii. Steepest Descent,
iv. Conjugate Gradient Method.

You need only give pseudocode for *one* of these methods. You may assume that the intial iterate, $\boldsymbol{x}^{(0)}$, is given as input to the algorithm. Your algorithm should calculate the k^{th} iterate, where k is input to the algorithm.

P4 (**20 pnts**) Let A be a symmetric, positive definite, $n \times n$ matrix, with $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ the discretization of an elliptic PDE. Consider the iterative method

$$\mathsf{Q}\boldsymbol{x}^{(k+1)} = (\mathsf{Q} - \omega\mathsf{A})\,\boldsymbol{x}^{(k)} + \omega\boldsymbol{b}.$$

(a) Show that

$$\boldsymbol{e}^{(k+1)} = \left(\mathsf{I} - \omega\mathsf{Q}^{-1}\mathsf{A}\right)\boldsymbol{e}^{(k)}.$$

(b) Show that a necessary and sufficient condition for convergence of this method is that the spectral radius of $\mathsf{I} - \omega\mathsf{Q}^{-1}\mathsf{A}$ be less than 1. (*Recall:* the spectral radius of a matrix B is the maximal value of $|\lambda|$ for λ an eigenvalue of B.)

P5 (**30 pnts**) Let A be a symmetric $n \times n$ matrix. We wish to solve the problem

$$\mathsf{A}\boldsymbol{x} = \boldsymbol{b}.$$

by using the iteration

$$\boldsymbol{x}^{(k+1)} = (\mathsf{I} - \omega\mathsf{A})\,\boldsymbol{x}^{(k)} + \omega\boldsymbol{b}.$$

Suppose that the eigenvalues of A are all in the interval $[3/4, 7/8]$. Let $\boldsymbol{e}^{(k)} = \boldsymbol{x}^{(k)} - \boldsymbol{x}$, where $\boldsymbol{x}$ solves the linear system.

(a) Find upper and lower bounds on the eigenvalues of $\mathsf{B} = \mathsf{I} - \omega\mathsf{A}$.

(b) Pick the ω such that the spectral radius of B is minimized.

(c) For your ω, give an upper bound on $\left\|\boldsymbol{e}^{(k+1)}\right\|_2 / \left\|\boldsymbol{e}^{(k)}\right\|_2$.

(d) Suppose we can find $\boldsymbol{x}^{(0)}$ such that $\left\|\boldsymbol{e}^{(0)}\right\|_2 \leq 1$. How many iterations are needed to obtain an approximate solution with accuracy of 1×10^{-10}? (*Hint:* if you have forgotten your calculator, you may use the fact that $\log_{10}(13)$ is very close to $10/9$.)

P6 (**10 pnts**) State some substantive question which you thought might appear on this exam, but did not. Answer this question (correctly).