

The final exam is Monday June 7th, 3pm-6pm. The final is a “one hour” exam covering the second half of the class, *i.e.*, solving elliptic PDE by iterative methods and the finite element method. There is also an optional “re-take” of the first exam, since many people did poorly on the first exam. This retake is also a one hour exam, covering general PDEs, and specific methods for solving parabolic and hyperbolic PDEs. If you elect to try the retake, you will have only the 3 hours to complete the final *and* the retake. If you do not wish to take the retake, you will have the full 3 hours to take the final.

You will not be penalized for trying the retake—I will consider the max of the two scores to be your first exam score. However, you can only improve your first exam score up to a 75. This means that there is no reason to try the retake if you scored better than 75 on the first exam, since you cannot improve your situation. In symbols, the grading scheme is as follows:

$$ex'_1 \leftarrow ex_1 \vee (retake \wedge 75) = \max \{ex_1, \min \{retake, 75\}\}.$$

What follows are good questions for the final exam, *i.e.*, the part covering the second half of the class. For more information on the retake, consult the original exam 1, and the exam 1 preparation sheet.

1. Give a discretization of the one dimensional elliptic “P”DE

$$\begin{cases} u_{xx} = f & x \in (0, 1), \\ u(0) = u_0 \\ u(1) = u_1 \end{cases}$$

Divide $[0, 1]$ into m pieces. By Taylor’s theorem, what is the size of the discretization error?

2. Consider the Dirichlet problem for *Poisson’s Equation*, *i.e.*,

$$\begin{cases} \nabla^2 u = u_{xx} + u_{yy} = f & (x, y) \in \mathcal{D}, \\ u(x, y) = \phi(x, y) & (x, y) \in \partial D. \end{cases}$$

Let v have the property that $\nabla^2 v$ is defined in \mathcal{D} , and that $v(x, y) = \phi(x, y)$ on ∂D .

Show that to solve the original problem, it suffices to find w such that

$$\begin{cases} \nabla^2 w = g =_{\text{df}} f - \nabla^2 v & (x, y) \in \mathcal{D}, \\ w(x, y) = 0 & (x, y) \in \partial D. \end{cases}$$

Then to let $u = w + v$.

(This shows that when considering the Dirichlet problem for Poisson’s Equation we can always assume that $\phi(x, y) = 0$.)

3. Let \mathbf{A} be a symmetric, positive definite, $n \times n$ matrix, with $\mathbf{Ax} = \mathbf{b}$ the discretization of an elliptic PDE. Consider the iterative method

$$\mathbf{Qx}^{(k+1)} = (\mathbf{Q} - \mathbf{A})\mathbf{x}^{(k)} + \mathbf{b}.$$

Identify the \mathbf{Q} such that this method is

- (a) Richardson's Method.
- (b) Jacobi iteration.
- (c) Gauss Seidel iteration.

Write out, in pseudo code, the Jacobi method, and the Gauss Seidel method.

4. What does it mean for \mathbf{v}, λ to be eigenvector and eigenvalue of matrix \mathbf{A} ?
- (a) Suppose $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of matrix \mathbf{A} . What are the eigenvalues of \mathbf{A}^{-1} .
 - (b) Suppose $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of matrix \mathbf{A} . What are the eigenvalues of $\mathbf{B} = p(\mathbf{A})$, where $p(x)$ is a polynomial?
 - (c) Suppose that the condition number of \mathbf{A} is 1, *i.e.*, that \mathbf{A} has a single distinct real nonzero eigenvalue, call it λ . Show that every nonzero vector is an eigenvector for \mathbf{A} . Argue that \mathbf{A} is $\lambda\mathbf{I}$.
5. Let \mathbf{v} be an n -dimensional vector, and let \mathbf{A} be an $n \times n$ matrix.
- (a) Define the norm $\|\mathbf{v}\|_2$.
 - (b) Define the norm $\|\mathbf{A}\|_2$.
 - (c) Show that $\|\mathbf{Av}\|_2 \leq \|\mathbf{A}\|_2 \|\mathbf{v}\|_2$.
 - (d) Show that $\|\mathbf{A}\|_2$ equals $\rho(\mathbf{A})$, the spectral radius of \mathbf{A} . (*Note:* the spectral radius of matrix \mathbf{A} , written $\rho(\mathbf{A})$, is its maximum eigenvalue in absolute value, *i.e.*, $\max_{\lambda_j} |\lambda_j|$ over all eigenvalues λ_j of \mathbf{A} .)
 - (e) Show that $\|\mathbf{A}^{-1}\|_2$ equals $(1/|\lambda_{\min}|)$, where λ_{\min} is the smallest, in absolute value, eigenvalue of \mathbf{A} .
 - (f) Define the condition number of a matrix, $\kappa(\mathbf{A})$. Show that $\kappa(\mathbf{A}) = \|\mathbf{A}\|_2 \|\mathbf{A}^{-1}\|_2$.
 - (g) Suppose there is some $\mu > 0$ such that, for a given \mathbf{A} ,

$$\|\mathbf{Av}\|_2 \geq \mu \|\mathbf{v}\|_2,$$

for all vectors \mathbf{v} .

- i. Show that $\mu \leq \|\mathbf{A}\|_2$. (Should be very simple.)
 - ii. Show that \mathbf{A} is nonsingular. (*Recall:* \mathbf{A} is singular if there is some $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{Ax} = \mathbf{0}$.)
 - iii. Show that $\|\mathbf{A}^{-1}\|_2 \leq (1/\mu)$.
 - (h) If \mathbf{A} is singular, is it necessarily the case that $\|\mathbf{A}\|_2 = 0$?
 - (i) If $\|\mathbf{A}\|_2 \geq \mu > 0$ does it follow that \mathbf{A} is nonsingular?
6. Let \mathbf{A} be a symmetric $n \times n$ matrix. We wish to solve the problem

$$\mathbf{Ax} = \mathbf{b}.$$

by using the iteration

$$\mathbf{x}^{(k+1)} = (\mathbf{I} - \mathbf{A})\mathbf{x}^{(k)} + \mathbf{b}.$$

Let $\mathbf{e}^{(k)} = \mathbf{x}^{(k)} - \mathbf{x}$, where \mathbf{x} solves the linear system.

- (a) Suppose $0.25 \leq \|e^{(k+1)}\|_2 / \|e^{(k)}\|_2 \leq 0.5$. What can you say about the eigenvalues of $B = I - A$? Then what can you say about the eigenvalues of A ?
- (b) What is the limit, as $k \rightarrow \infty$, of $x^{(k)}$? (*Hint*: what is the limit of $e^{(k)}$?)
- (c) Suppose we start with $x^{(0)} = \mathbf{0}$. How many iterations are needed to obtain an approximate solution with accuracy of 1×10^{-6} ?
- (d) Can you pick an ω such that the “extrapolated” iteration

$$x^{(k+1)} = (I - \omega A)x^{(k)} + \omega b$$

gives better convergence rates? For your ω what is the upper bound on $\|e^{(k+1)}\|_2 / \|e^{(k)}\|_2$?

7. Let A be a symmetric positive definite $n \times n$ matrix with n distinct eigenvalues. Letting $y^{(0)} = b / \|b\|_2$, consider the iteration

$$y^{(k+1)} = \frac{Ay^{(k)}}{\|Ay^{(k)}\|_2}.$$

- (a) What is $\|y^{(k)}\|_2$?
- (b) Show that $y^{(k)} = A^k b / \|A^k b\|_2$.
- (c) Show that as $k \rightarrow \infty$, $y^{(k)}$ converges to the (normalized) eigenvector associated with the largest eigenvalue of A .
8. Let A be a symmetric, positive definite, $n \times n$ matrix, with $Ax = b$ the discretization of an elliptic PDE. Consider the iterative method

$$Qx^{(k+1)} = (Q - \omega A)x^{(k)} + \omega b.$$

Show that

$$e^{(k+1)} = (I - \omega Q^{-1}A)e^{(k)}.$$

- (a) Show that a necessary and sufficient condition for convergence of this method is that the spectral radius of $I - \omega Q^{-1}A$ be less than 1.
- (b) Supposing the eigenvalues of $Q^{-1}A$ are in $[\alpha, \beta]$ with $0 < \alpha$, where are the eigenvalues of $I - \omega Q^{-1}A$? Pick ω to minimize the spectral radius of $I - \omega Q^{-1}A$.
9. Let A be a symmetric, positive definite, $n \times n$ matrix.
- (a) Define the A energy norm, $\|v\|_A$.
- (b) What does it mean for a set of vectors, $\{v_i\}_{i=1}^n$ to be A -orthonormal?
10. Let A be a symmetric, positive definite, $n \times n$ matrix, with $Ax = b$ the discretization of an elliptic PDE.
- (a) Show that solving this linear equation is equivalent to minimizing

$$f(x) = \frac{1}{2}x^\top Ax - x^\top b.$$

- (b) For given x, v , define $g(t) = f(x + tv)$. Using freshman calculus, find the t that minimizes $g(t)$. (*Note*: you will use the fact that A is positive definite.)
- (c) Describe, in pseudo code, the *steepest descent* iterative method for minimizing this quadratic form.

- (d) Presupposing the existence of a collection of \mathbf{A} -orthonormal vectors, $\{\mathbf{v}_i\}_{i=1}^n$, describe, in pseudo code, the *conjugate directions* iterative method for minimizing this quadratic form.

11. Consider the one dimensional elliptic “P”DE

$$\begin{cases} (xu')' + u = x & x \in (0, 1), \\ u'(0) = 1 \\ u(1) = 0 \end{cases}$$

You will consider the weak solution to this problem. Let

$$\mathcal{H} = \left\{ v: \mathbb{R} \rightarrow \mathbb{R} \mid \int_0^1 |v''|^2 dx < \infty, v(1) = 0 \right\}$$

- (a) Multiply the PDE by $v \in \mathcal{H}$, and take the integral over $[0, 1]$.
 (b) Use integration by parts to expand $(xu')'v$. Simplify this expression assuming that $u, v \in \mathcal{H}$.
 (c) Simplify your equation to get something like

$$\int_0^1 (? - ?) dx = \int_0^1 xvdx.$$

- (d) Let $\mathcal{S}_h \subseteq \mathcal{H}$ be the subset of all polynomial functions of \mathcal{H} of degree no greater than 2.
 i. Argue that $\phi_1(x) = x - 1$, and $\phi_2(x) = x^2 - 1$ are in \mathcal{S}_h .
 ii. Prove that $\mathcal{S}_h = \text{span}\{\phi_1, \phi_2\}$. (*Hint*: A typical element of \mathcal{S}_h is of the form $v(x) = ax^2 + bx + c$. Use this and the fact that $v(1) = 0$ to express v as the linear combination of ϕ_1, ϕ_2 .)
 iii. Evaluate the following:

$$\begin{array}{ccc} \int_0^1 x\phi_1(x)dx & \int_0^1 x\phi_2(x)dx & \\ \int_0^1 \phi_1(x)\phi_1(x)dx & \int_0^1 \phi_2(x)\phi_2(x)dx & \int_0^1 \phi_1(x)\phi_2(x)dx \\ \int_0^1 x\phi_1'(x)\phi_1'(x)dx & \int_0^1 x\phi_2'(x)\phi_2'(x)dx & \int_0^1 x\phi_1'(x)\phi_2'(x)dx \end{array}$$

- (e) Let $u = u_1\phi_1 + u_2\phi_2$. Setting $v = \phi_1$, and $v = \phi_2$, use your integral evaluations above to rewrite your integral as a linear system:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- (f) Conclude that $u_1 = 1, u_2 = 0$, *i.e.*, that the best approximate weak solution to the PDE in the space \mathcal{S}_h is the function $\phi_1(x) = x - 1$.
 (g) Verify that $u = \phi_1$ actually solves the given PDE.