Homework 4

April 23, 2004

This homework is due Friday, April 30. Remember that the **first exam** is Monday, May 3, at 5pm, in HSS 1305, which I believe is next door to the regular classroom.

1. Do problems 4.1 and 4.3 from §4.6 of the notes [1].
2. Prove that the Richardson difference scheme is unstable. See §3.5 of the notes [1]. The Richardson scheme uses a centered difference in time to attempt to solve the heat equation:

$$\frac{U_n^{j+1} - U_n^{j-1}}{2k} = \frac{U_{n-1}^j - 2U_n^j + U_{n+1}^j}{h^2} + f_n^j$$

   Use the Fourier Analysis Method:
   (a) Assume that the error Z_n^j follows the same recurrence relation, but for the homogeneous problem.
   (b) Pretend that $Z_n^j = \lambda^j e^{i\beta nh}$, where $i = \sqrt{-1}$, and λ, β are to be found. Plug this into the recurrence to get a quadratic equation involving λ. Use the fact that $1 - \cos\alpha = 2\sin^2(\alpha/2)$
   (c) Show that one of the roots is "too big," *i.e.*, that $|\lambda_1| > 1$. *Hint:* show the roots are real, and that that the discriminant $b^2 - 4ac$ is too large.
   (d) Argue that the Richardson scheme is unstable.
3. ([2] *ex.* #2.6)
   (a) Consider the quadratic equation $z^2 + bz + c = 0$, with real coefficients b, c. Show that the roots of this equation are in or on the unit disk (*i.e.*, are complex numbers with modulus less than or equal to one) if and only if $|c| \leq 1$, and $|b| \leq 1 + c$ *Hint:* When the roots are complex, you should assume that they are conjugates.
   (b) By Fourier Analysis and the above fact, show the following scheme is stable:

$$\frac{U_n^{j+1} - U_n^{j-1}}{2k} = \frac{1}{3h^2}\left(U_{n-1}^{j+1} - 2U_n^{j+1} + U_{n+1}^{j+1} + U_{n-1}^j - 2U_n^j + U_{n+1}^j + U_{n-1}^{j-1} - 2U_n^{j-1} + U_{n+1}^{j-1}\right)$$

References

[1] Randolph E. Bank, Peter Rentrop and Donald R. Smith. Numerical treatment of partial differential equations, 1995. Course Notes for UCSD Math M172.

[2] K. W. Morton and D. F. Mayers. *Numerical solution of partial differential equations*. Cambridge University Press, Cambridge, 1994. ISBN 0-521-41855-0; 0-521-42922-6. An introduction.