Homework 6

May 18, 2004

This homework is due Friday, May 28.

1. Let A be a symmetric $n \times n$ matrix. We wish to solve the problem

$$\mathsf{A}\boldsymbol{x} = \boldsymbol{b}.$$

   by using the iteration

$$\boldsymbol{x}^{(k+1)} = (\mathsf{I} - \mathsf{A})\,\boldsymbol{x}^{(k)} + \boldsymbol{b}.$$

   (This is known as Richardson's Method.) Let $\boldsymbol{e}^{(k)} = \boldsymbol{x}^{(k)} - \boldsymbol{x}$, where $\boldsymbol{x}$ solves the system.
   (a) Suppose $\left\|\boldsymbol{e}^{(k+1)}\right\|_2 \leq 0.2\left\|\boldsymbol{e}^{(k)}\right\|_2$. What can you say about the eigenvalues of $\mathsf{B} = \mathsf{I} - \mathsf{A}$? Then what can you say about the eigenvalues of A?
   (b) What is the limit, as $k \to \infty$, of $\boldsymbol{x}^{(k)}$? (*Hint:* what is the limit of $\boldsymbol{e}^{(k)}$?)
   (c) Suppose we start with $\boldsymbol{x}^{(0)} = \mathbf{0}$. How many iterations are needed to obtain an approximate solution with accuracy of 1×10^{-6}?
2. Let A be a symmetric $n \times n$ matrix with eigenvalues in the interval $[\alpha, \beta]$, with $0 < \alpha \leq \beta$. Consider the iteration

$$\boldsymbol{x}^{(k+1)} = (\mathsf{I} - \omega\mathsf{A})\,\boldsymbol{x}^{(k)} + \omega\boldsymbol{b},$$

   with $\omega \neq 0$ some constant. Let $\boldsymbol{e}^{(k)} = \boldsymbol{x}^{(k)} - \boldsymbol{x}$, where $\boldsymbol{x}$ solves the system.
   (a) Show that $\boldsymbol{e}^{(k+1)} = (\mathsf{I} - \omega\mathsf{A})\,\boldsymbol{e}^{(k)}$.
   (b) Show that the eigenvalues of $\mathsf{I} - \omega\mathsf{A}$ are in the interval $[1 - \omega\beta, 1 - \omega\alpha]$.
   (c) Prove that

$$\max\left\{|\lambda| : 1 - \omega\beta \leq \lambda \leq 1 - \omega\alpha\right\}$$

   is minimized when we choose ω such that $1 - \omega\beta = -(1 - \omega\alpha)$. (*Hint:* it may help to look at the graph of something versus ω.)
   (d) Pick the ω that satisfies this relationship. For this ω find a bound on

$$\left\|\boldsymbol{e}^{(k+1)}\right\|_2 / \left\|\boldsymbol{e}^{(k)}\right\|_2.$$

3. Let A be a nonsingular $n \times n$ matrix. We wish to solve $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$. Let $\boldsymbol{x}^{(0)}$ be some starting vector, let $\mathcal{D}_k$ be $\operatorname{span}\left\{\boldsymbol{r}^{(0)}, \mathsf{A}\boldsymbol{r}^{(0)}, \ldots, \mathsf{A}^k\boldsymbol{r}^{(0)}\right\}$, and let $\mathcal{P}_k$ be the set of polynomials, $p(x)$ of degree k with $p(0) = 1$.
   Consider the iterative method: Let $\boldsymbol{x}^{(k+1)}$ be the $\boldsymbol{x}$ that solves

$$\min_{\boldsymbol{x} \in \boldsymbol{x}^{(0)} + \mathcal{D}_k} \|\boldsymbol{b} - \mathsf{A}\boldsymbol{x}\|_2.$$

   Let $\boldsymbol{r}^{(k)} = \boldsymbol{b} - \mathsf{A}\boldsymbol{x}^{(k)}$.
   (a) Show that if $\boldsymbol{x} \in \boldsymbol{x}^{(0)} + \mathcal{D}_k$, then $\boldsymbol{b} - \mathsf{A}\boldsymbol{x} = p(\mathsf{A})\boldsymbol{r}^{(0)}$ for some $p \in \mathcal{P}_k$.

(b) Prove that, conversely, for any $p \in \mathcal{P}_k$ there is some $\boldsymbol{x} \in \boldsymbol{x}^{(0)} + \mathcal{D}_k$, such that $\boldsymbol{b} - \mathsf{A}\boldsymbol{x} = p(\mathsf{A})\boldsymbol{r}^{(0)}$.

(c) Argue that

$$\left\| \boldsymbol{r}^{(k+1)} \right\|_2 = \min_{p \in \mathcal{P}_k} \left\| p(\mathsf{A}) \boldsymbol{r}^{(0)} \right\|_2 .$$

(d) Prove that this iteration converges in at most n steps. (*Hint:* Argue for the existence of a polynomial in $\mathcal{P}_n$ that vanishes at all the eigenvalues of A. Use this polynomial to show that $\left\| \boldsymbol{r}^{(n)} \right\|_2 \leq 0$.)