Exam 2 Preparation

The following formulæ will be provided on your exam:

$$s = \int \left|\boldsymbol{R}'(t)\right| \mathrm{d}t \qquad \nabla f = \langle f_x, f_y \rangle \qquad \mathrm{D}_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u} \qquad D = f_{xx} f_{yy} - f_{yx} f_{xy}$$

Everything else must be committed to memory.

What follows are questions similar to your homework assignments. You should be prepared to answer at least all these questions. In no way should you consider this list exhaustive.

1. Find the arc length of a parametrized curve, *i.e.*, given a curve parametrized by $\boldsymbol{R}(t), a \leq t \leq b$, find the arc length of the curve. *e.g.*,:
   (a) $\boldsymbol{R}(t) = \langle 2\cos t, 2\sin t \rangle, 0 \leq t \leq \pi$.
   answer: 2π.
   (b) $\boldsymbol{R}(t) = \left\langle \sqrt{2}t, e^t, e^{-t} \right\rangle, 0 \leq t \leq 1$.
   answer: $e^1 - e^{-1}$.
   (c) $\boldsymbol{R}(t) = \left\langle t^2, t, 3 \right\rangle, 0 \leq t \leq 1$.
   answer: The integral might be tricky. I get $\frac{\sqrt{5}}{2} + \frac{1}{4}\operatorname{arcsinh} 2$.
2. Given the position of a particle, $\boldsymbol{R}(t)$, find its velocity, speed, and acceleration. *e.g.*,:
   (a) $\boldsymbol{R}(t) = \langle 2\cos t, 2\sin t \rangle$.
   answer: $\boldsymbol{v}(t) = \langle -2\sin t, 2\cos t \rangle$, $s(t) = 2$, $\boldsymbol{a}(t) = \langle -2\cos t, -2\sin t \rangle$.
   (b) $\boldsymbol{R}(t) = \left\langle \sqrt{2}t, e^t, e^{-t} \right\rangle$.
   answer: $\boldsymbol{v}(t) = \left\langle \sqrt{2}, e^t, -e^{-t} \right\rangle$, $s(t) = e^t + e^{-t}$, $\boldsymbol{a}(t) = \left\langle 0, e^t, e^{-t} \right\rangle$.
   (c) $\boldsymbol{R}(t) = \left\langle t^2, t, 3 \right\rangle$.
   answer: $\boldsymbol{v}(t) = \langle 2t, 1, 0 \rangle$, $s(t) = \sqrt{4t^2 + 1}$, $\boldsymbol{a}(t) = \langle 2, 0, 0 \rangle$.
3. Given the acceleration of a particle and its initial position and velocity, find the position of the particle. *e.g.*,:
   (a) $\boldsymbol{a}(t) = \langle 1, t, \sin t \rangle, \boldsymbol{v}(0) = \langle 0, 1, 0 \rangle, \boldsymbol{R}(0) = \langle 0, 0, 0 \rangle$.
   answer: $\boldsymbol{R}(t) = \left\langle \frac{1}{2}t^2, \frac{1}{6}t^3 + t, t - \sin t \right\rangle$.
   (b) $\boldsymbol{a}(t) = \langle \sin t, \cos t \rangle, \boldsymbol{v}(0) = \langle 1, 0 \rangle, \boldsymbol{R}(0) = \langle 0, 1 \rangle$.
   answer: $\boldsymbol{R}(t) = \langle 2t - \sin t, 2 - \cos t \rangle$.
4. (this is a basic one) You should understand functions of two or more variables. *e.g.*,: given $f(x, y)$, be able to evaluate $f(1, 2)$; find the domain of f; sketch a graph of $f(x, y)$, or the contours (level sets) of $f(x, y)$.
   (a) What is the domain of $f(x, y) = \frac{\sqrt{x-y}}{y}$?
   answer: The points (x, y) with $x \geq y$, and $y \neq 0$.
   (b) Graph the level sets of $f(x, y) = \frac{1}{4}x^2 + y^2$.
   answer: These are ellipses.
   (c) Graph the function $f(x, y) = \sqrt{x^2 + y^2}$.
   answer: A paraboloid.
   (d) Graph the function and level sets of $f(x, y) = 3x - 4y + 2$.
   answer: The graph is the plane $-3x + 4y + z = 2$, the level sets are lines.
5. You should be able to find the limit of a function of two variables, or recognize that the limit does not exist. You need to know what functions are continuous, and how this helps you evaluate a limit. *e.g.*, evaluate:

(a) $\lim\limits_{(x,y)\to(1,1)} \frac{x^6 y}{2x^2 - y}$.

answer: By continuity, this is 1.

(b) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 - y^2}{4 + x^2 + \cos y}$,

answer: By continuity, this is 0.

(c) $\lim\limits_{(x,y)\to(0,0)} \frac{x-y}{x+y}$,

answer: Does not exist

(d) $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$,

answer: Does not exist

(e) $\lim\limits_{(x,y)\to(0,0)} \frac{6x^3 y}{2x^4 + y^4}$.

answer: Does not exist

(f) $\lim\limits_{(x,y)\to(0,0)} \frac{xy + x^2 y^2}{xy^2 + x^2 y}$.

answer: This one has some domain problems. First try $x = y > 0$, with $x \to 0$. Then try $x = y < 0$ with $x \to 0$. You should get $\infty, -\infty$ respectively. Thus the limit does not exist. This is a considerably harder problem than the rest of them.

6. Given a function $f(x, y)$, find the partials $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$, etc. When do we have $f_{xy} = f_{yx}$?

7. Given a function $f(x, y)$, find the gradient ∇f. *e.g.*,:

(a) $f(x, y) = x^2 y + y$.

answer: $\nabla f = \langle 2xy, x^2 + 1 \rangle$.

(b) $f(x, y) = \cos(xy) + x^2 + y^2$.

answer: $\nabla f = \langle -\sin(xy)\, y + 2x, -\sin(xy)\, x + 2y \rangle$.

8. Given a function $f(x, y)$, and a unit vector $\boldsymbol{u}$, find the directional derivative of f in the direction of $\boldsymbol{u}$, $\mathrm{D}_{\boldsymbol{u}} f(x, y)$. *e.g.*,:

(a) $f(x, y) = x^2 + xy + y^2, \boldsymbol{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

answer: $2x + \frac{11}{5} y$.

(b) $f(x, y) = \cos\frac{x}{y} + \cos\frac{y}{x}, \boldsymbol{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

answer: $\frac{1}{\sqrt{2}} \left(-\frac{1}{y} \sin\frac{x}{y} + \frac{y}{x^2} \sin\frac{y}{x} - \frac{1}{x} \sin\frac{y}{x} + \frac{x}{y^2} \sin\frac{x}{y} \right)$.

(c) $f(x, y) = \arctan xy, \boldsymbol{u} = \langle 1, 0 \rangle$, at the point $(x, y) = (1, 1)$.

answer: $\frac{1}{2}$.

9. Understand the connection between ∇f and the directional derivative, the direction of maximal increase of f, and the maximal rate of instantaneous increase of f. *e.g.*,:

(a) For $f(x, y) = \sqrt{x^2 + y^2}$ at $(3, 4)$, what is the direction and rate of maximal increase of f?

answer: $\langle 3/5, 4/5 \rangle$, and the rate is 1.

10. Given a function $f(x, y)$, and point (x_0, y_0), find the equation of the tangent plane to f at this point. *e.g.*,:

(a) $f(x, y) = x^2 + 2xy + y^2, (x_0, y_0) = (1, 1)$.

answer: $z - 4 = 4(x - 1) + 4(y - 1)$.

(b) $f(x, y) = \cos(xy^2), (x_0, y_0) = (\pi, 1)$.

answer: $z = -1$.

11. Given a function $f(x, y)$, and point (x_0, y_0), find the equation of the linearization of

f, call it $L(x,y)$ at the given point. Use the linearization to approximate $f(x,y)$ for some point $(x,y) \approx (x_0,y_0)$. *e.g.,*:

(a) $f(x,y) = (x+y)^3, (x_0,y_0) = (1,1)$, approximate $f(0.9,1.1)$.
answer: $L(x,y) = 8 + 12(x-1) + 12(y-1)$. $L(0.9,1.1) = 8$.

(b) $f(x,y) = \sqrt{xy}, (x_0,y_0) = (2,8)$, approximate $f(2.1,8.1)$.
answer: $L(x,y) = 4 + (x-2) + \frac{1}{4}(y-8) = x + \frac{1}{4}y$. $L(2.1,8.1) = 4.125$.

12. Be prepared to use the chain rule to find derivatives and partial derivatives. *e.g.,*:

(a) $f(x,y) = (x+y)^4, x = \sqrt{t}, y = t^2$, find $\frac{\mathrm{df}}{\mathrm{dt}}$.
answer: $4\left(\sqrt{t}+t^2\right)\left[\frac{1}{2\sqrt{t}} + 2t\right]$.

(b) $f(x,y) = x^2 + \cos(xy), x = s+t, y = s^2$, find $\frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$.
answer: $\frac{\partial f}{\partial s} = \nabla f \cdot \langle 1, 2s\rangle, \frac{\partial f}{\partial t} = \nabla f \cdot \langle 1, 0\rangle$.

(c) $f(x,y,z) = \sqrt{x^2+y^2+z^2}, x = s\cos t, y = s\sin t, z = s^2$, find $\frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$.
answer: $\frac{\partial f}{\partial s} = \frac{1}{\sqrt{s^2+s^4}}\left(s+2s^3\right), \frac{\partial f}{\partial t} = 0$.

13. Given a function $f(x,y)$, be prepared to find all its critical points. Be able to apply the second derivative test to see if you have a max, min, or neither. Be able to find extremal values of $f(x,y)$ on a closed, bounded domain D. *e.g.,*:

(a) $f(x,y) = x^4 + y^4 - 4xy$.
answer: Local minima at $(-1,-1)$ and $(1,1)$. Saddle point at $(0,0)$.

(b) $f(x,y) = (1+xy)(x+y)$.

(c) $f(x,y) = e^x \cos y$.
answer: This has no critical points. Thus no extrema on the unbounded set R^2.

(d) $f(x,y) = x + 3y - 4$ on the triangle with corners $(0,0)$,$(1,0)$,$(0,1)$.
answer: This has no critical points. Min of -4 at $(0,0)$, and max of -1 at $(0,1)$.

(e) $f(x,y) = 2x^2 + 3y^2 - 4x - 5$ on the disc $x^2 + y^2 \leq 16$.
answer: I covered this in class.

14. Given a function $f(x,y)$, find the extremal values of f subject to the constraint $g(x,y) = k$ by the method of Lagrange Multipliers.

(a) $f(x,y) = x + xy + y$, subject to $x^2 + y^2 = 9$.
answer: At $\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$, takes min value $\frac{9}{2} - \frac{6}{\sqrt{2}}$. At $\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, takes max value $\frac{9}{2} + \frac{6}{\sqrt{2}}$.

(b) $f(x,y) = \frac{1}{xy}$, subject to $x + y = 4$.
answer: This is a poor Lagrange question, but we expect a local min of $\frac{1}{4}$ at $(2,2)$.

(c) $f(x,y,z) = 8x - 4z$, subject to $x^2 + 10y^2 + z^2 = 5$.
answer: At $(-2,1)$ takes min value -20. At $(2,-1)$ takes max value 20

(d) $f(x,y,z) = xyz$, subject to $2xz + 2yz + xy = 12$.
answer: I did something like this in class.