Final Exam Preparation

The following formulæ will be provided on your exam:

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3 \qquad \boldsymbol{a}\times\boldsymbol{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \qquad L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\mathrm{d}A = \mathrm{d}y\mathrm{d}x = r\mathrm{d}r\mathrm{d}\theta \quad s = \int \left|\boldsymbol{R}'(t)\right| \mathrm{d}t \qquad \mathrm{D}_{\boldsymbol{u}}f = \nabla f\cdot\boldsymbol{u} \qquad \nabla f = \langle f_x, f_y\rangle$$

$$D = f_{xx}f_{yy} - f_{yx}f_{xy} \qquad \nabla f = \lambda\nabla g.$$

Everything else must be committed to memory.

What follows are questions similar to your homework assignments. You should be prepared to answer at least all these questions. In no way should you consider this list exhaustive.

Beware that I have not yet tried to answer some of these questions, and they may be challenging or impossible. If you know how to answer the question in general and get stuck on algebra or something does not seem right, the problem may be bogus. These inconsistencies may be corrected if and when I get around to answering these questions.

1. Let $\boldsymbol{a} = \langle 1,0,1\rangle$, $\boldsymbol{b} = \langle 1,3,0\rangle$.
   (a) Find $|\boldsymbol{a}|$.
   (b) Find a unit vector in the same direction as $\boldsymbol{a}$.
   (c) Find $3\boldsymbol{a} - 2\boldsymbol{b}$.
   (d) Find $\boldsymbol{a}\cdot\boldsymbol{b}$.
   (e) Find $\boldsymbol{a}\times\boldsymbol{b}$.
   (f) Find a vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.
   (g) Find the angle subtended by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
   (h) Find the component of $\boldsymbol{a}$ along $\boldsymbol{b}$.
   (i) Find the projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.
   (j) Give a parametrization of the line through the point $(3,4,1)$ parallel to $\boldsymbol{b}$.
   (k) Give a parametrization of the plane through the origin that is parallel to both $\boldsymbol{a}$ and $\boldsymbol{b}$.
   (l) Give a parametrization of the plane containing the point $(1,2,-1)$ that has $\boldsymbol{a}$ as a normal.
2. Let $\boldsymbol{R_1}(t) = \langle 0,3,1\rangle + t\langle 2,1,-1\rangle$ parametrize the line l_1. Let $\boldsymbol{R_2}(t) = \langle 1,2,1\rangle + t\langle 2,3,0\rangle$ parametrize the line l_2. Let $3x - 2y + 5z = 1$ describe a plane P.
   (a) Find a vector parallel to l_1. Find a vector parallel to l_2.
   (b) Find a vector normal to P.
   (c) Find the intersection of the two lines l_1, l_2.
   (d) Find the intersection of l_1 and P.
   (e) Find the intersection of l_2 and P. (*Hint:* it's a trick)
   (f) Find the angle subtended by l_1, l_2.
   (g) Find the angle that l_1 subtends with the plane P.
   (h) Find the distance from l_1 to the point $(2,2,1)$.
   (i) Find the distance from l_2 to P.
3. Find the limits:

(a) $\lim_{t\to 0} \boldsymbol{R}(t)$, where $\boldsymbol{R}(t) = \left\langle t+1, 5, t^2 \right\rangle$.
(b) $\lim_{t\to 0} \left\langle \frac{\sin t}{t}, e^{-t}, t \right\rangle$.
(c) $\lim_{t\to 0} \left\langle \frac{t^2+t^4}{4t^2-t^3}, 5t+4, \ln(1+t) \right\rangle$.

4. Find $\boldsymbol{R}'(t)$ for
   (a) $\boldsymbol{R}(t) = \left\langle t+1, 5, t^2 \right\rangle$.
   (b) $\boldsymbol{R}(t) = \langle 4, 7, 1 \rangle \times \left\langle t+1, 5, t^2 \right\rangle$.
   (c) $\boldsymbol{R}(t) = \langle \arctan t, \sin t, \cos t \rangle$.
5. Given $\boldsymbol{R}(t)$, parametrizing the position of a particle at time t, find the velocity, speed, and acceleration of the particle. Find the arc length of the curve parametrized by $\boldsymbol{R}(t)$ for $0 \le t \le 1$, for
   (a) $\boldsymbol{R}(t) = \langle 3, -1, 4 \rangle + t \langle 12, 3, 4 \rangle$.
   (b) $\boldsymbol{R}(t) = \langle 3\cos t, 3\sin t, 4t \rangle$.
   (c) $\boldsymbol{R}(t) = \left\langle \sqrt{2}t, e^t, e^{-t} \right\rangle$.
   (d) $\boldsymbol{R}(t) = \left\langle \frac{t^2}{2}, 2, 2t \right\rangle$. (*Hint:* might be hard. try trig integral?)
6. Given the acceleration of a particle and its initial position and velocity, find the position of the particle. *e.g.*,:
   (a) $\boldsymbol{a}(t) = \left\langle t^2, -2t, \cos t \right\rangle, \boldsymbol{v}(0) = \langle 0, 0, 0 \rangle, \boldsymbol{R}(0) = \langle 1, 2, 1 \rangle$.
   (b) $\boldsymbol{a}(t) = \langle 3, 3\sin t, 3\cos t \rangle, \boldsymbol{v}(0) = \langle 1, 0, 0 \rangle, \boldsymbol{R}(0) = \langle 0, 0, 1 \rangle$.
7. Given $f(x, y)$, find the domain of f, evaluate $f(1, 3)$, sketch the graph of f or its contours.
   (a) $f(x, y) = 3x + 2y - 5$.
   (b) $f(x, y) = 4 - x^2 - y^2$.
   (c) $f(x, y) = \sqrt{x^2 + y^2}$.
   (d) $f(x, y) = \frac{\sqrt{y-x+2}}{\ln y}$; (Do not try to graph or find contours)
8. Find the limits, or show they do not exist:
   (a) $\lim_{(x,y)\to(1,1)} \frac{x+y}{2x-y}$.
   (b) $\lim_{(x,y)\to(2,1)} \frac{x^3}{3+x^2-y}$.
   (c) $\lim_{(x,y)\to(0,0)} \frac{x+y}{1+x^2+y^2}$.
   (d) $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$,
   (e) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$,
   (f) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^4-y^4}$,
   (g) $\lim_{(x,y)\to(0,0)} \frac{x^4+y^2}{2x^4+y^4}$,

   The last two are harder.
9. Given $f(x, y)$, find ∇f, and find partials f_{xx}, f_{xy}, f_{yy} :
   (a) $f(x, y) = 3x^2 + yx$.
   (b) $f(x, y) = \sin(xy)$.
   (c) $f(x, y) = (xy + \cos x)^3$.
10. Given a function $f(x, y)$, and a unit vector $\boldsymbol{u}$, find the directional derivative of f in the direction of $\boldsymbol{u}$, $\mathrm{D}_{\boldsymbol{u}} f(x, y)$. *e.g.*,:

(a) $f(x,y) = (x-y)^2, \boldsymbol{u} = \left\langle \frac{-3}{5}, \frac{4}{5} \right\rangle$.
(b) $f(x,y) = x + xy + y^2, \boldsymbol{u} = \langle 0, 1 \rangle$.
(c) $f(x,y) = \sin(xy), \boldsymbol{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, at the point $(x,y) = (0,\pi)$.

11. Given $f(x,y)$ and a point (x_0,y_0), find the direction and rate of maximal increase of f at (x_0,y_0). Find the plane tangent to f at (x_0,y_0); find the linearization $L(x,y)$ of f at (x_0,y_0). *e.g.*,:
    (a) $f(x,y) = x^2 + y^2, (x_0,y_0) = (1,0)$.
    (b) $f(x,y) = (x-y)^2, (x_0,y_0) = (-1,-1)$.
    (c) $f(x,y) = 3 + xy^2, (x_0,y_0) = (1,1)$.
    (d) $f(x,y) = \sin(xy), (x_0,y_0) = (0,\pi)$.
12. Use the linearization of $f(x,y)$ about (x_0,y_0) to approximate $f(x,y)$ at (x,y) near (x_0,y_0). *e.g.*,:
    (a) With $f(x,y) = (x-y)^2, (x_0,y_0) = (1,1)$, approximate $(1.1 - 0.8)^2$.
    (b) With $f(x,y) = \sqrt[3]{x+y}, (x_0,y_0) = (2,6)$, approximate $\sqrt[3]{2.1 + 6.1}$.
13. Use the chain rule to find derivatives and partial derivatives. *e.g.*,:
    (a) $f(x,y) = (x+y)^2, x = e^t, y = \sqrt{t}$, find $\frac{\mathrm{d}f}{\mathrm{d}t}$.
    (b) $f(x,y) = \sin(xy), x = st, y = t^2 + s^2$, find $\frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$.
    (c) $f(x,y,z) = \left(x^2 + y^2 + z^2\right)^3, x = s\cos t, y = s\sin t, z = s$, find $\frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$.
14. Given a function $f(x,y)$, find all its critical points in R^2. Apply the second derivative test to see if each critical point is a max, min, or neither. *e.g.*,:
    (a) $f(x,y) = \left(x^2 + y^2\right)^2$.
    (b) $f(x,y) = \left(x^2 - y^2\right)^2$.
    (c) $f(x,y) = (xy)(x+y)$.
    (d) $f(x,y) = (1 - x - y)\left(x^2 + y\right)$.
15. Given a function $f(x,y)$ and a closed bounded set D, find the extremal values of f on D. Do this by first finding the critical points inside D, then finding the extremal values of f on the boundary of D; in the case that the boundary is simple, this can be done by substitution; if the boundary of D is a level curve of a function $g(x,y)$, use the method of Lagrange Multipliers. *e.g.*,:
    (a) $f(x,y) = x + xy + y$, D is the triangle with corners $(0,0)$, $(3,0)$, $(0,2)$.
    (b) $f(x,y) = \frac{1}{4+2x+y}$, D is the disc $x^2 + y^2 \leq 1$.
    (c) $f(x,y) = xy$, D is the ellipse $x^2 + 4y^2 \leq 4$.
    (d) $f(x,y,z) = 2x + y - z$, D is the ball $x^2 + y^2 + z^2 \leq 9$.
16. Given a function $f(x,y)$ and a rectangle R, evaluate

$$\iint\limits_R f(x,y)\mathrm{d}A$$

    (a) $f(x,y) = x + y, R = [0,1] \times [2,3]$.
    (b) $f(x,y) = x\sin y, R = [0,1] \times [0,\pi]$.
    (c) $f(x,y) = x/y, R = [0,1] \times [1,2]$.
    (d) $f(x,y) = \cos(x + 2y), R = [0,\pi] \times [0,\pi/2]$.

17. Given a function $f(x,y)$ and a general region in the plane, D, evaluate

$$\iint_D f(x,y)\mathrm{d}A$$

(a) $f(x,y) = x + y, D = \left\{(x,y) \,\middle|\, x^2 + y^2 \leq 4\right\}$. (actually this is easy by symmetry considerations.)
(b) $f(x,y) = x - y$, D is the triangle with corners $(0,0)$, $(0,5)$, $(3,2)$.
(c) $f(x,y) = \sqrt{1-x^2}$, D is the triangle with corners $(0,0)$, $(1,0)$, $(1,1)$.
(d) $f(x,y) = (x-y)^2$, D is the region between the curves $y = x^2$ and $y = x$.

18. Given a function $f(x,y)$ and a general region in the plane, D, interpret the integral

$$\iint_D f(x,y)\mathrm{d}A$$

as a polar integral and solve. *Note:* you should be able to recognize when it is appropriate or beneficial to evaluate a double integral in polar coordinates. *e.g.,*:
(a) $f(x,y) = e^{x^2+y^2}, D = \left\{(x,y) \,\middle|\, x^2 + y^2 \leq 9\right\}$.
(b) $f(x,y) = 2, D = \left\{(x,y) \,\middle|\, x^2 + y^2 \leq 1,\ y \geq x\right\}$.
(c) $f(x,y) = x + y, D = \left\{(x,y) \,\middle|\, y \geq 0,\ 4 \leq x^2 + y^2 \leq 9\right\}$.
(d) $f(x,y) = \sqrt{x^2+y^2}, D = \{(r,\theta) \mid 0 \leq \theta \leq \pi/4,\ 0 \leq r \leq \cos 2\theta\}$.

19. Given a density function $\rho(x,y)$ for a lamina in the plane, D, find the center of mass of the lamina. *e.g.,*:
(a) $\rho(x,y) = x + y + 2, D = \left\{(x,y) \,\middle|\, x^2 + y^2 \leq 1\right\}$.
(b) $\rho(x,y) = 2, D = \left\{(x,y) \,\middle|\, 0 \leq x \leq 1,\ 0 \leq y \leq x^2\right\}$.
(c) $\rho(x,y) = \frac{1}{\sqrt{x^2+y^2}}, D = \left\{(x,y) \,\middle|\, 0 \leq x,\ 1 \leq x^2 + y^2 \leq 4\right\}$. (*Hint:* try converting to polar integrals?)

20. Given a function $f(x,y,z)$, and some general region B in three space, evaluate

$$\iiint_B f(x,y,z)\mathrm{d}V$$

(a) $f(x,y,z) = 1 + xy,\ B = \{(x,y,z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 2,\ 1 \leq z \leq 2\}$.
(b) $f(x,y,z) = yz\cos\left(x^5\right),\ B = \{(x,y,z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x,\ x \leq z \leq 2x\}$. (This is §15.7 #8)
(c) $f(x,y,z) = xz,\ B = \{(x,y,z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x,\ 0 \leq z \leq 6 - x - y\}$.

21. On the real exam you may not be told what technique is appropriate for a given problem; you must decide this for yourself. In this category are the following miscellaneous problems:
(a) Find the volume of the tetrahedron with corners $(0,0,0)$, $(2,0,0)$, $(0,1,0)$, $(0,0,3)$.
(b) Find numbers a, b, c with $a + b + c = 100$, such that a^2bc is a maximum.
(c) Let $p(x,y,z) = x^2y + z^2x$ represent the density of plankton in the ocean. A whale is located at the point (() 1,2,0). In which direction should the whale move to maximize the increase in plankton concentration?