Final Exam Preparation

The following formulæ will be provided on your exam:

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3 \qquad \boldsymbol{a}\times\boldsymbol{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \qquad L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\mathrm{d}A = \mathrm{d}y\mathrm{d}x = r\mathrm{d}r\mathrm{d}\theta \quad s = \int \left|\boldsymbol{R}'(t)\right| \mathrm{d}t \qquad \mathrm{D}_{\boldsymbol{u}}f = \nabla f\cdot\boldsymbol{u} \qquad \nabla f = \langle f_x, f_y\rangle$$

$$D = f_{xx}f_{yy} - f_{yx}f_{xy} \qquad \nabla f = \lambda\nabla g.$$

Everything else must be committed to memory.

What follows are questions similar to your homework assignments. You should be prepared to answer at least all these questions. In no way should you consider this list exhaustive.

1. Let $\boldsymbol{a} = \langle 1,0,1\rangle$, $\boldsymbol{b} = \langle 1,3,0\rangle$.
   (a) Find $|\boldsymbol{a}|$.
   answer: $\sqrt{2}$.
   (b) Find a unit vector in the same direction as $\boldsymbol{a}$.
   answer: $\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\rangle$.
   (c) Find $3\boldsymbol{a} - 2\boldsymbol{b}$.
   answer: $\langle 1,-6,3\rangle$.
   (d) Find $\boldsymbol{a}\cdot\boldsymbol{b}$.
   answer: 1.
   (e) Find $\boldsymbol{a}\times\boldsymbol{b}$.
   answer: $\langle -3,1,3\rangle$.
   (f) Find a vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.
   answer: $\langle -3,1,3\rangle$.
   (g) Find the angle subtended by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
   answer: $\arccos\left(\frac{1}{2\sqrt{5}}\right)$.
   (h) Find the component of $\boldsymbol{a}$ along $\boldsymbol{b}$.
   answer: $\frac{1}{\sqrt{10}}$.
   (i) Find the projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.
   answer: $\left\langle \frac{1}{10}, \frac{3}{10}, 0\right\rangle$.
   (j) Give a parametrization of the line through the point $(3,4,1)$ parallel to $\boldsymbol{b}$.
   answer: $\langle 3,4,1\rangle + t\langle 1,3,0\rangle$.
   (k) Give a parametrization of the plane through the origin that is parallel to both $\boldsymbol{a}$ and $\boldsymbol{b}$.
   answer: $-3x + y + 3z = 0$.
   (l) Give a parametrization of the plane containing the point $(1,2,-1)$ that has $\boldsymbol{a}$ as a normal.
   answer: $x + z = 0$.
2. Let $\boldsymbol{R_1}(t) = \langle 0,3,1\rangle + t\langle 2,1,-1\rangle$ parametrize the line l_1. Let $\boldsymbol{R_2}(t) = \langle 1,2,1\rangle + t\langle 2,3,0\rangle$ parametrize the line l_2. Let $3x - 2y + 5z = 1$ describe a plane P.
   (a) Find a vector parallel to l_1. Find a vector parallel to l_2.

answer: $\langle 2,1,-1\rangle, \langle 2,3,0\rangle$.

(b) Find a vector normal to P.

answer: $\langle 3,-2,5\rangle$.

(c) Find the intersection of the two lines l_1, l_2.

answer: They do not intersect.

(d) Find the intersection of l_1 and P.

answer: $(-4,1,3)$.

(e) Find the intersection of l_2 and P. (*Hint:* it's a trick)

answer: They do not intersect.

(f) Find the angle subtended by l_1, l_2.

answer: Ooops! They do not intersect, so this question is more or less bogus. I intended for you to find the angle between the vectors $\langle 2,1,-1\rangle$ and $\langle 2,3,0\rangle$, which you do using the formula for dot product.

(g) Find the angle that l_1 subtends with the plane P.

answer: $\arccos\left(\frac{1}{2\sqrt{57}}\right)$.

(h) Find the distance from l_1 to the point $(2,2,1)$.

answer: $\sqrt{7/2}$.

(i) Find the distance from l_2 to P.

answer: $\sqrt{2/19}$.

3. Find the limits:

(a) $\lim_{t\to 0} \boldsymbol{R}(t)$, where $\boldsymbol{R}(t) = \left\langle t+1, 5, t^2\right\rangle$.

answer: $\langle 1,5,0\rangle$.

(b) $\lim_{t\to 0} \left\langle \frac{\sin t}{t}, e^{-t}, t\right\rangle$.

answer: $\langle 1,1,0\rangle$.

(c) $\lim_{t\to 0} \left\langle \frac{t^2+t^4}{4t^2-t^3}, 5t+4, \ln(1+t)\right\rangle$.

answer: $\left\langle \frac{1}{4}, 4, 0\right\rangle$.

4. Find $\boldsymbol{R}'(t)$ for

(a) $\boldsymbol{R}(t) = \left\langle t+1, 5, t^2\right\rangle$.

answer: $\langle 1,0,2t\rangle$.

(b) $\boldsymbol{R}(t) = \langle 4,7,1\rangle \times \left\langle t+1,5,t^2\right\rangle$.

answer: $\langle 4,7,1\rangle \times \langle 1,0,2t\rangle = \langle 14t, 1-8t, -7\rangle$.

(c) $\boldsymbol{R}(t) = \langle \arctan t, \sin t, \cos t\rangle$.

answer: $\left\langle \frac{1}{1+t^2}, \cos t, -\sin t\right\rangle$.

5. Given $\boldsymbol{R}(t)$, parametrizing the position of a particle at time t, find the velocity, speed, and acceleration of the particle. Find the arc length of the curve parametrized by $\boldsymbol{R}(t)$ for $0 \le t \le 1$, for

(a) $\boldsymbol{R}(t) = \langle 3,-1,4\rangle + t\langle 12,3,4\rangle$.

answer: $\boldsymbol{R}'(t) = \langle 12,3,4\rangle$, $|\boldsymbol{R}'(t)| = 13$, $\boldsymbol{R}''(t) = \langle 0,0,0\rangle$, arc length is 13.

(b) $\boldsymbol{R}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$.

answer: $\boldsymbol{R}'(t) = \langle -3\sin t, 3\cos t, 4\rangle$, $|\boldsymbol{R}'(t)| = 5$, $\boldsymbol{R}''(t) = \langle -3\cos t, -3\sin t, 0\rangle$, arc length is 5.

(c) $\boldsymbol{R}(t) = \left\langle \sqrt{2}t, e^t, e^{-t}\right\rangle$.

answer: $\boldsymbol{R}'(t) = \left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle$, $|\boldsymbol{R}'(t)| = e^t + e^{-t}$, $\boldsymbol{R}''(t) = \left\langle 0, e^t, e^{-t}\right\rangle$, arc length is $e - e^{-1}$.

(d) $\boldsymbol{R}(t) = \left\langle \frac{t^2}{2}, 2, 2t \right\rangle$. (*Hint:* might be hard. try trig integral?)

answer: $\boldsymbol{R}'(t) = \langle t, 0, 2\rangle$, $|\boldsymbol{R}'(t)| = \sqrt{t^2+4}$, $\boldsymbol{R}''(t) = \langle 1, 0, 0\rangle$, arc length is $2\sqrt{5} + 8\,\text{arcsinh}\,\frac{1}{2}$.

6. Given the acceleration of a particle and its initial position and velocity, find the position of the particle. *e.g.*,:

(a) $\boldsymbol{a}(t) = \langle t^2, -2t, \cos t\rangle$, $\boldsymbol{v}(0) = \langle 0,0,0\rangle$, $\boldsymbol{R}(0) = \langle 1,2,1\rangle$.

answer: $\left\langle \frac{1}{12}t^4 + 1, -\frac{1}{3}t^3 + 2, 2 - \cos t \right\rangle$.

(b) $\boldsymbol{a}(t) = \langle 3, 3\sin t, 3\cos t\rangle$, $\boldsymbol{v}(0) = \langle 1,0,0\rangle$, $\boldsymbol{R}(0) = \langle 0,0,1\rangle$.

answer: $\left\langle \frac{3}{2}t^2 + t, 3t - 3\sin t, 4 - 3\cos t \right\rangle$.

7. Given $f(x,y)$, find the domain of f, evaluate $f(1,3)$, sketch the graph of f or its contours.

(a) $f(x,y) = 3x + 2y - 5$.

answer: Domain is all reals, $f(1,3) = 4$, the graph is a plane, the contours are parallel straight lines of the form $y = -\frac{3}{2}x + k$.

(b) $f(x,y) = 4 - x^2 - y^2$.

answer: Domain is all reals, $f(1,3) = -6$, the graph is a paraboloid with a global maximum at $(0,0,4)$, the contours are concentric circles centered around the origin.

(c) $f(x,y) = \sqrt{x^2+y^2}$.

answer: Domain is all reals, $f(1,3) = \sqrt{10}$, the graph is a cone with a global minimum at $(0,0,0)$, the contours are concentric circles centered around the origin.

(d) $f(x,y) = \frac{\sqrt{y-x+2}}{\ln y}$; (Do not try to graph or find contours)

answer: The domain is $y > 0, y \neq 1, y \geq x - 2$, $f(1,3) = \frac{2}{\ln 3}$.

8. Find the limits, or show they do not exist:

(a) $\lim_{(x,y)\to(1,1)} \frac{x+y}{2x-y}$.

answer: The function is continuous at $(1,1)$, so the answer is $f(1,1) = 2$.

(b) $\lim_{(x,y)\to(2,1)} \frac{x^3}{3+x^2-y}$.

answer: The function is continuous at $(2,1)$, so the answer is $f(2,1) = \frac{8}{6}$.

(c) $\lim_{(x,y)\to(0,0)} \frac{x+y}{1+x^2+y^2}$.

answer: The function is continuous at $(0,0)$, so the answer is $f(0,0) = 0$.

(d) $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$,

answer: The limit does not exist: With $x = 0$ taking $y \to 0$ gives a limit of -1, while fixing $y = 0$, and taking $x \to 0$ gives a limit of 1. They do not match, so the limit does not exist.

(e) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$,

answer: The limit does not exist: Taking $x = y \to 0$ gives a limit of $\frac{1}{2}$, while taking $x = -y \to 0$ gives a limit of $-\frac{1}{2}$. They do not match, so the limit does not exist.

(f) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^4-y^4}$, *answer*: On its domain, this function is equal to $\frac{1}{x^2+y^2}$, which tends to infinity as $(x,y) \to (0,0)$. This is not a good question for this class.

(g) $\lim_{(x,y)\to(0,0)} \frac{x^4+y^2}{2x^4+y^4}$,

answer: The limit does not exist: With $y=0$ taking $x \to 0$ gives a limit of $\frac{1}{2}$, while fixing $x=0$, and taking $y \to 0$ gives a limit of ∞. They do not match, so the limit does not exist. This is not a good question for this class.

The last two are harder.

9. Given $f(x,y)$, find ∇f, and find partials f_{xx}, f_{xy}, f_{yy} :
   (a) $f(x,y) = 3x^2 + yx$.
   answer: $\nabla f(x,y) = \langle 6x+y, x \rangle$, $f_{xx} = 6, f_{xy} = 1, f_{yy} = 0$.
   (b) $f(x,y) = \sin(xy)$.
   answer: I had incorrect answers here before–the mixed partial f_{xy} was wrong. Here's the correct versions: $\nabla f(x,y) = \langle y\cos(xy), x\cos(xy) \rangle$, $f_{xx} = -y^2 \sin(xy)$, $f_{xy} = \cos xy - yx \sin(xy)$, $f_{yy} = -x^2 \sin(xy)$.
   (c) $f(x,y) = (xy + \cos x)^3$.
   answer: I completely dropped the ball on the original answers to this one. Here's the correct versions: $\nabla f(x,y) = \left\langle 3(xy+\cos x)^2 [y - \sin x], 3(xy+\cos x)^2 x \right\rangle$,
   $f_{xx} = 6(xy+\cos x)[y-\sin x]^2 + 3(xy+\cos x)^2(-\cos x)$,
   $f_{xy} = 6(xy+\cos x)x[y-\sin x] + 3(xy+\cos x)^2$, $f_{yy} = 6(xy+\cos x)x^2$.
10. Given a function $f(x,y)$, and a unit vector $\boldsymbol{u}$, find the directional derivative of f in the direction of $\boldsymbol{u}$, $\mathrm{D}_{\boldsymbol{u}} f(x,y)$. *e.g.*,:
    (a) $f(x,y) = (x-y)^2$, $\boldsymbol{u} = \langle \frac{-3}{5}, \frac{4}{5} \rangle$.
    answer: $\frac{-14}{5}(x-y)$.
    (b) $f(x,y) = x + xy + y^2$, $\boldsymbol{u} = \langle 0, 1 \rangle$.
    answer: $x + 2y$.
    (c) $f(x,y) = \sin(xy)$, $\boldsymbol{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, at the point $(x,y) = (0,\pi)$.
    answer: $\frac{\pi}{\sqrt{2}}$.
11. Given $f(x,y)$ and a point (x_0,y_0), find the direction and rate of maximal increase of f at (x_0,y_0). Find the plane tangent to f at (x_0,y_0); find the linearization $L(x,y)$ of f at (x_0,y_0). *e.g.*,:
    (a) $f(x,y) = x^2 + y^2$, $(x_0,y_0) = (1,0)$.
    answer: Direction is $\langle 2,0 \rangle$, rate is 2, $L(x,y) = 1 + 2(x-1)$
    (b) $f(x,y) = (x-y)^2$, $(x_0,y_0) = (-1,-1)$.
    answer: Direction is $\langle 0,0 \rangle$, rate is 0, $L(x,y) = 0$
    (c) $f(x,y) = 3 + xy^2$, $(x_0,y_0) = (1,1)$.
    answer: Direction is $\langle 1,2 \rangle$, rate is $\sqrt{5}$, $L(x,y) = 4 + (x-1) + 2(y-1)$
    (d) $f(x,y) = \sin(xy)$, $(x_0,y_0) = (0,\pi)$.
    answer: Direction is $\langle \pi,0 \rangle$, rate is π, $L(x,y) = 0 + \pi(x-0)$
12. Use the linearization of $f(x,y)$ about (x_0,y_0) to approximate $f(x,y)$ at (x,y) near (x_0,y_0). *e.g.*,:
    (a) With $f(x,y) = (x-y)^2$, $(x_0,y_0) = (1,1)$, approximate $(1.1-0.8)^2$.
    answer: Oooops! $L(x,y) = 0$, so the approximate value is $L(1.1, 0.8) = 0$, not a good approximation.

(b) With $f(x,y) = \sqrt[3]{x+y}, (x_0, y_0) = (2,6)$, approximate $\sqrt[3]{2.1+6.1}$.
answer: $L(x,y) = 2 + \frac{1}{12}(x-2) + \frac{1}{12}(y-6)$, so the approximate value is $L(2.1, 6.1) = 2 + \frac{1}{60}$.

13. Use the chain rule to find derivatives and partial derivatives. *e.g.,*:
   (a) $f(x,y) = (x+y)^2, x = e^t, y = \sqrt{t}$, find $\frac{\mathrm{d}f}{\mathrm{d}t}$.
   answer: $\frac{\mathrm{d}f}{\mathrm{d}t} = 2\left(e^t + \sqrt{t}\right)\left[e^t + \frac{1}{2\sqrt{t}}\right]$
   (b) $f(x,y) = \sin(xy), x = st, y = t^2 + s^2$, find $\frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$.
   answer: $\frac{\mathrm{d}f}{\mathrm{d}s} = \cos\left(st\left[s^2+t^2\right]\right)\left[3s^2t + t^3\right]$, $\frac{\mathrm{d}f}{\mathrm{d}t} = \cos\left(st\left[s^2+t^2\right]\right)\left[3t^2s + s^3\right]$,
   (c) $f(x,y,z) = \left(x^2+y^2+z^2\right)^3, x = s\cos t, y = s\sin t, z = s$, find $\frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$.
   answer: $\frac{\mathrm{d}f}{\mathrm{d}s} = 48s^5$, $\frac{\mathrm{d}f}{\mathrm{d}t} = 0$.

14. Given a function $f(x,y)$, find all its critical points in R^2. Apply the second derivative test to see if each critical point is a max, min, or neither. *e.g.,*:
   (a) $f(x,y) = \left(x^2+y^2\right)^2$.
   answer: Critical points: $(0,0)$, a local (and global!) minimum.
   (b) $f(x,y) = \left(x^2-y^2\right)^2$.
   answer: Critical points: $(0,0)$, neither a min nor max.
   (c) $f(x,y) = (xy)(x+y)$.
   answer: Critical points: $(0,0)$, second derivative test is inconclusive (*i.e.*, we get $D = 0$).
   (d) $f(x,y) = (1-x-y)\left(x^2+y\right)$.
   answer: Critical points: $\left(\frac{1}{2}, \frac{1}{8}\right)$, which is a saddle point, neither a local max or min.

15. Given a function $f(x,y)$ and a closed bounded set D, find the extremal values of f on D. Do this by first finding the critical points inside D, then finding the extremal values of f on the boundary of D; in the case that the boundary is simple, this can be done by substitution; if the boundary of D is a level curve of a function $g(x,y)$, use the method of Lagrange Multipliers. *e.g.,*:
   (a) $f(x,y) = x + xy + y$, D is the triangle with corners $(0,0)$, $(3,0)$, $(0,2)$.
   answer: There are no critical points in D. The maximum value is at $(3,0)$ where f takes value 3; the minimum is at $(0,0)$, with value 0.
   (b) $f(x,y) = \frac{1}{4+2x+y}$, D is the disc $x^2+y^2 \leq 1$.
   answer: There are no critical points in D. The maximum value is at $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$, where f takes value $4+\sqrt{5}$; The minimum value is at $\left(\frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$, where f takes value $4-\sqrt{5}$;
   (c) $f(x,y) = xy$, D is the ellipse $x^2+4y^2 \leq 4$.
   answer: There is a critical point $(0,0)$ in D, but it is a saddle point by the second derivative test. By Lagrange Multiplier Method, the maximum value of $\sqrt{2}$ is achieved at the points $\left(\sqrt{2}, \frac{\sqrt{2}}{2}\right)$, and $\left(-\sqrt{2}, \frac{-\sqrt{2}}{2}\right)$, while the minimum value of $-\sqrt{2}$ is achieved at the points $\left(-\sqrt{2}, \frac{\sqrt{2}}{2}\right)$, and $\left(\sqrt{2}, \frac{-\sqrt{2}}{2}\right)$.
   (d) $f(x,y,z) = 2x + y - z$, D is the ball $x^2+y^2+z^2 \leq 9$.
   answer: There are no critical points of f. By Lagrange Multiplier Method, f

takes a maximum value of $2\left(\sqrt{6}+\sqrt{3/2}\right)$, at the point $\left(\sqrt{6},\sqrt{3/2},-\sqrt{3/2}\right)$, while it takes its minimum value of $-2\left(\sqrt{6}+\sqrt{3/2}\right)$, at the point $\left(-\sqrt{6},-\sqrt{3/2},\sqrt{3/2}\right)$.

16. Given a function $f(x,y)$ and a rectangle R, evaluate

$$\iint\limits_R f(x,y)\mathrm{d}A$$

(a) $f(x,y)=x+y, R=[0,1]\times[2,3]$.
answer: $\int_0^1\int_2^3 x+y\ \mathrm{d}y\mathrm{d}x=3$.
(b) $f(x,y)=x\sin\,y, R=[0,1]\times[0,\pi]$.
answer: $\int_0^1\int_0^\pi x\sin\,y\ \mathrm{d}y\mathrm{d}x=1$.
(c) $f(x,y)=x/y, R=[0,1]\times[1,2]$.
answer: $\int_0^1\int_1^2 x/y\ \mathrm{d}y\mathrm{d}x=\frac{1}{2}\ln 2$.
(d) $f(x,y)=\cos\left(x+2y\right), R=[0,\pi]\times[0,\pi/2]$.
answer: $\int_0^\pi\int_0^{\pi/2}\cos(x+2y)\ \mathrm{d}y\mathrm{d}x=-2$.

17. Given a function $f(x,y)$ and a general region in the plane, D, evaluate

$$\iint\limits_D f(x,y)\mathrm{d}A$$

(a) $f(x,y)=x+y, D=\left\{(x,y)\,\middle|\,x^2+y^2\leq 4\right\}$. (actually this is easy by symmetry considerations.)
answer: $\int_0^2\int_0^{2\pi}\left(r\cos(\theta)+r\sin(\theta)\right)\,r\mathrm{d}\theta\mathrm{d}r=0$.
(b) $f(x,y)=x-y$, D is the triangle with corners $(0,0)$, $(0,5)$, $(3,2)$.
answer: $\int_0^3\int_{\frac{2x}{3}}^{5-x} x-y\ \mathrm{d}y\mathrm{d}x=-10$.
(c) $f(x,y)=\sqrt{1-x^2}$, D is the triangle with corners $(0,0)$, $(1,0)$, $(1,1)$.
answer: $\int_0^1\int_0^x\sqrt{1-x^2}\ \mathrm{d}y\mathrm{d}x=\int_0^1 x\sqrt{1-x^2}\mathrm{d}x=\frac{1}{3}$.
(d) $f(x,y)=(x-y)^2$, D is the region between the curves $y=x^2$ and $y=x$.
answer: $\int_0^1\int_{x^2}^x (x-y)^2\ \mathrm{d}y\mathrm{d}x=\int_0^1 x\sqrt{1-x^2}\mathrm{d}x=\frac{1}{420}$?

18. Given a function $f(x,y)$ and a general region in the plane, D, interpret the integral

$$\iint\limits_D f(x,y)\mathrm{d}A$$

as a polar integral and solve. *Note:* you should be able to recognize when it is appropriate or beneficial to evaluate a double integral in polar coordinates. *e.g.*,:
(a) $f(x,y)=e^{x^2+y^2}, D=\left\{(x,y)\,\middle|\,x^2+y^2\leq 9\right\}$.
answer: $\int_0^3\int_0^{2\pi} e^{r^2}\ r\mathrm{d}\theta\mathrm{d}r=\pi\left(e^9-1\right)$.
(b) $f(x,y)=2, D=\left\{(x,y)\,\middle|\,x^2+y^2\leq 1,\ y\geq x\right\}$.
answer: $\int_0^1\int_{\pi/4}^{5\pi/4} 2\,r\mathrm{d}\theta\mathrm{d}r=\pi$.
(c) $f(x,y)=x+y, D=\left\{(x,y)\,\middle|\,y\geq 0,\ 4\leq x^2+y^2\leq 9\right\}$.
answer: $\int_2^3\int_0^\pi r\left(\cos\theta+\sin\theta\right)\,r\mathrm{d}\theta\mathrm{d}r=\frac{46}{3}$.
(d) $f(x,y)=\sqrt{x^2+y^2}, D=\{(r,\theta)\,|\,0\leq\theta\leq\pi/4,\ 0\leq r\leq\cos 2\theta\}$.
answer: $\int_0^{\pi/4}\int_0^{\cos 2\theta} r\,r\mathrm{d}r\mathrm{d}\theta=\frac{1}{9}$.

19. Given a density function $\rho(x, y)$ for a lamina in the plane, D, find the center of mass of the lamina. *e.g.,*:
   (a) $\rho(x, y) = x + y + 2, D = \left\{(x, y) \,\middle|\, x^2 + y^2 \leq 1\right\}$.
   answer: First we find the mass: $\iint_D \rho(x, y)\mathrm{d}A = 2\pi$. Then $\bar{x} = \frac{1}{m}\iint_D x\rho(x, y)\mathrm{d}A = \frac{1}{8}$. Similarly $\bar{y} = \frac{1}{m}\iint_D y\rho(x, y)\mathrm{d}A = \frac{1}{8}$. So the center of mass is $\left(\frac{1}{8}, \frac{1}{8}\right)$.
   (b) $\rho(x, y) = 2, D = \left\{(x, y) \,\middle|\, 0 \leq x \leq 1,\, 0 \leq y \leq x^2\right\}$.
   answer: First we find the mass: $\iint_D \rho(x, y)\mathrm{d}A = \frac{2}{3}$. Then $\bar{x} = \frac{1}{m}\iint_D x\rho(x, y)\mathrm{d}A = \frac{3}{4}$. Similarly $\bar{y} = \frac{1}{m}\iint_D y\rho(x, y)\mathrm{d}A = \frac{3}{10}$. So the center of mass is $\left(\frac{3}{4}, \frac{3}{10}\right)$.
   (c) $\rho(x, y) = \frac{1}{\sqrt{x^2+y^2}}, D = \left\{(x, y) \,\middle|\, 0 \leq x,\, 1 \leq x^2 + y^2 \leq 4\right\}$. (*Hint:* try converting to polar integrals?)
   answer: First we find the mass: $\iint_D \rho(x, y)\mathrm{d}A = \int_1^2 \int_{-\pi/2}^{\pi/2} \frac{1}{r} r\mathrm{d}\theta\mathrm{d}r = \pi$. $\bar{x} = \frac{1}{m}\iint_D x\rho(x, y)\mathrm{d}A = \frac{3}{\pi}$. Similarly $\bar{y} = \frac{1}{m}\iint_D y\rho(x, y)\mathrm{d}A = 0$. So the center of mass is $\left(\frac{3}{\pi}, 0\right)$.
20. Given a function $f(x, y, z)$, and some general region B in three space, evaluate

$$\iiint\limits_B f(x, y, z)\mathrm{d}V$$

   (a) $f(x, y, z) = 1 + xy,\ B = \{(x, y, z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 2,\ 1 \leq z \leq 2\}$.
   answer: $\int_0^1 \int_0^2 \int_1^2 1 + xy\,\mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_0^1 \int_0^2 1 + xy\,\mathrm{d}y\mathrm{d}x = \int_0^1 2 + 2x\,\mathrm{d}x = 3$.
   (b) $f(x, y, z) = yz\cos\left(x^5\right),\ B = \{(x, y, z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x,\ x \leq z \leq 2x\}$. (This is §15.7 #8)
   answer: $\int_0^1 \int_0^x \int_x^{2x} yz\cos x^5\,\mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_0^1 \int_0^x \frac{3}{2}x^2 y\cos x^5\,\mathrm{d}y\mathrm{d}x = \int_0^1 \frac{3}{4}x^4\cos x^5\,\mathrm{d}x = \frac{3}{20}\sin 1$.
   (c) $f(x, y, z) = xz,\ B = \{(x, y, z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x,\ 0 \leq z \leq 6 - x - y\}$.
   answer: $\int_0^1 \int_0^x \int_0^{6-x-y} xz\,\mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_0^1 \int_0^x \frac{x}{2}(6 - x - y)\,\mathrm{d}y\mathrm{d}x = \int_0^1 3x^2 - \frac{3}{4}x^3\,\mathrm{d}x = \frac{13}{16}$.
21. On the real exam you may not be told what technique is appropriate for a given problem; you must decide this for yourself. In this category are the following miscellaneous problems:
   (a) Find the volume of the tetrahedron with corners $(0, 0, 0)$, $(2, 0, 0)$, $(0, 1, 0)$, $(0, 0, 3)$.
   answer: This is $\int_0^2 \int_0^{1-\frac{x}{2}} \int_0^{\frac{6-3x-6y}{2}} 1\,\mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_0^2 \int_0^{1-\frac{x}{2}} \frac{6-3x-6y}{2}\,\mathrm{d}y\mathrm{d}x = \int_0^2 \frac{3}{2} - \frac{3x}{2} + \frac{3x^2}{8}\,\mathrm{d}x = 1$.
   (b) Find numbers a, b, c with $a + b + c = 100$, such that a^2bc is a maximum.
   answer: Maximize the function $f(a, b) = a^2b(100 - a - b)$. Find that it has a maximum at $a = 50, b = 25$. Then $c = 25$.
   (c) Let $p(x, y, z) = x^2y + z^2x$ represent the density of plankton in the ocean. A whale is located at the point $(1, 2, 0)$. In which direction should the whale move to maximize the increase in plankton concentration?
   answer: The gradient is the direction of maximal increase. In this case we have $\nabla p(x, y, z) = \left\langle 2xy + z^2, x^2, 2zx\right\rangle$. At the point $(1, 2, 0)$ this takes value $\langle 4, 1, 0\rangle$. To denote a direction, we unitize this vector to get $\frac{1}{\sqrt{17}}\langle 4, 1, 0\rangle$.