Exam 2 Preparation

The following formulæ will be provided on your exam:

$$\mathrm{d}\boldsymbol{R} = \boldsymbol{R}'(t)\mathrm{d}t \quad \mathrm{d}\boldsymbol{S} = \left(\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}x} \times \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}y}\right)\mathrm{d}x\mathrm{d}y \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Everything else must be committed to memory.

What follows are questions similar to your homework assignments. You should be prepared to answer at least all these questions. In no way should you consider this list exhaustive.

1. Find the "integral of the tangential component" of vector field along a curve. *e.g.*,:
   (a) $\boldsymbol{F} = \left\langle 3x+4y, 2x+3y^2 \right\rangle$, along $x^2+y^2=4$. (this is a closed curve)
   (b) $\boldsymbol{F} = \left\langle 2xy, x^2+z, y \right\rangle$ along the straight line from $(1,0,2)$ to $(3,4,1)$.
   (c) $\boldsymbol{F} = \langle yz, xz, xy \rangle$ along *any* curve between $(-1,-1,2)$ to $(-2,1,-1)$.
   (d) $\boldsymbol{F} = \left\langle 2xz + y^2z^2, 2xyz^2, x^2+2xy^2z \right\rangle$ along the curve parametrized by $\boldsymbol{R}(t) = \left\langle \cos(\pi t) + e^t, -\sin(\pi t) + \log(1+t), t^{12} \right\rangle$ from $(2,0,0)$ to $(e-1, \log 2, 1)$.
2. Given a vector field $\boldsymbol{F}$, determine if it is conservative. If it is conservative, find ϕ such that $\boldsymbol{F} = \nabla\phi$. *e.g.*,
   (a) $\boldsymbol{F} = \left\langle 2xy, x^2+z, y \right\rangle$.
   (b) $\boldsymbol{F} = \langle xy, z, x \rangle$.
   (c) $\boldsymbol{F} = \left\langle x^2y+1, \frac{1}{3}x^3+1, y \right\rangle$.
   (d) $\boldsymbol{F} = \langle -\sin x \cos y, \cos x \cos y, 1 \rangle$.
   (e) $\boldsymbol{F} = \frac{1}{x^2+y^2+z^2} \langle x, y, z \rangle$.
   (f) $\boldsymbol{F} = \frac{1}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle$.
3. Give two conditions equivalent to $\boldsymbol{F}$ being a conservative field.
4. If $\boldsymbol{F}$ is conservative, what can we say about $\nabla \times \boldsymbol{F}$? Under what conditions does the converse hold? Why do we care if $\boldsymbol{F}$ is conservative? What does it allow us to do?
5. Suppose $\nabla\phi = \nabla\psi$. What more can we say about ϕ and ψ?
6. Given some parametrized surface, S, find $\mathrm{d}\boldsymbol{S}$ and $\mathrm{d}S$ in terms of $\mathrm{d}u$ and $\mathrm{d}v$. *e.g.*,:
   (a) S is parametrized by $\boldsymbol{R}(u,v) = \left\langle v\cos u, v\sin u, v^2 \right\rangle$.
   (b) S is parametrized by $\boldsymbol{R}(u,v) = \langle (1+\cos u)\cos v, (1+\cos u)\sin v, \sin v \rangle$.
7. Given some surface, S, find a parametrization $\boldsymbol{R}(u,v)$ of S. Find the appropriate limits for u, v. Find $\mathrm{d}\boldsymbol{S}$ and $\mathrm{d}S$ in terms of $\mathrm{d}u$ and $\mathrm{d}v$. Set up an integral for the surface area of S. *e.g.*,:
   (a) S is the part of the cone $x^2 = y^2 + z^2$ inside the sphere $(x-8)^2 + y^2 + z^2 = 25$.
   (b) S is the part of the paraboloid $z = x^2 + y^2$ above the xy plane and below the plane $x+y+z=9$.
8. Given some surface, S, and a vector field $\boldsymbol{F}$ find the integral of the flux:

$$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}.$$

   e.g.,:
   (a) $\boldsymbol{F} = \langle 0, 0, \cos(xy + 2^z) \rangle$, S is the part of the cylinder $x^2+y^2=1$ with $0 \le z \le 2$.
   (b) $\boldsymbol{F} = \langle x, y, -z \rangle$, S is the surface of the ellipsoid $x^2 + \frac{1}{4}y^2 + \frac{1}{9}z^2 = 9$.

(c) $\boldsymbol{F} = \langle x^2, y, z^3 \rangle$, S is the surface of the cube bounded by $x = 0, x = 2, y = \pm 1, z = \pm 1$.

(d) $\boldsymbol{F} = \langle y + z, 2x + y, y \rangle$, S is the surface of the triangle with corners $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$ with normal pointing away from the origin.

(e) $\boldsymbol{F} = \langle 2y, 2x, z \rangle$, S is the part of the cone $x^2 = y^2 + z^2$ inside the sphere $(x - 8)^2 + y^2 + z^2 = 25$. Assume $\boldsymbol{n} \cdot \boldsymbol{i} < 0$.

(f) $\boldsymbol{F} = \langle 3, x^2, y \rangle$, S is the part of the paraboloid $z = x^2 + y^2$ above the xy plane and below the plane $x + y + z = 9$. Assume $\vec{n} \cdot \boldsymbol{k} > 0$.

You should be prepared to answer questions of this type directly. You should also be able to recognize when the Divergence Theorem applies, and be able to apply it.

9. State the Divergence Theorem. State Stokes' Theorem.

10. Compute a "volume" integral *i.e.*, compute

$$\iint_R f(x, y)\mathrm{d}A \quad \text{or} \quad \iiint_R f(x, y, z)\mathrm{d}V$$

by converting to (and evaluating) an iterated integral:

$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x, y)\mathrm{d}y\mathrm{d}x \quad \text{or} \quad \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

(a) $f(x, y) = 2xy$, R is bounded by $y = 0, x = 2, y = x^2$.

(b) $f(x, y) = y^2 \cos x$, R is bounded by $x = y^3, y = -1, y = 1, x = 3$.

(c) $f(x, y, z) = \sqrt{x^2 + z^2}$, R is bounded by $y = x^2 + z^2, y = 4$.

(d) $f(x, y, z) = x + y + z$, R is bounded by $x = y^2, x = z, z = 0, z = 1$.

You should be able to effortlessly convert to polar, cylindrical, or spherical coordinates and correctly add the appropriate integrating factors.

11. You should be able to compute a "volume" integral by making a change of variables. You should be able to compute the Jacobian of the transformation. You should be able to compute the Jacobian for an "implicit" transformation (one in which u, v are given in terms of x, y and not the other way around). You should be able to figure out what the transformed region, $\tilde{R}$ looks like, and how to figure out its extents.

(a) $f(x, y) = \sin\left(x^2 + 2xy + y^2\right)$, R is the triangle with corners $(0, 0)$,$(1, 0)$,$(0, 1)$ using the change $u = x + y, v = x - y$.

(b) $f(x, y) = e^{xy}$, R is bounded by $xy = 1, xy = 4, y = 1, y = 3$ using the change $x = u/v, y = v$.

(c) $f(x, y, z) = \frac{2y}{x+2y+4z}$, R is bounded by $1 \le x + 2y + 4z \le 4$, $0 \le y - 2z \le 1$, $0 \le y + 2z \le 2$, using an affine transform.

(d) $f(x, y) = (x - y)^3$, R is bounded by $y = x, y = 3x, x = 3/2$, using the transform $u = x - y, v = 3x - y$.

(e) $f(x, y, z) = \frac{1}{\sqrt{x^2 + \frac{1}{9}y^2 + \frac{1}{16}z^2}}$, with R the ellipsoid $x^2 + \frac{1}{9}y^2 + \frac{1}{16}z^2 \le 25$.