Exam 2 Preparation

The following formulæ will be provided on your exam:

$$\mathrm{d}\boldsymbol{R} = \boldsymbol{R}'(t)\mathrm{d}t \quad \mathrm{d}\boldsymbol{S} = \left(\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}x} \times \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}y}\right)\mathrm{d}x\mathrm{d}y \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Everything else must be committed to memory.

What follows are questions similar to your homework assignments. You should be prepared to answer at least all these questions. In no way should you consider this list exhaustive.

1. Find the "integral of the tangential component" of vector field along a curve. *e.g.*,:
   (a) $\boldsymbol{F} = \left\langle 3x + 4y, 2x + 3y^2 \right\rangle$, along $x^2 + y^2 = 4$. (this is a closed curve)
   answer: A tedious calculation involving trigonometry.
   (b) $\boldsymbol{F} = \left\langle 2xy, x^2 + z, y \right\rangle$ along the straight line from $(1, 0, 2)$ to $(3, 4, 1)$.
   answer: 40. Can be done directly or by using a potential.
   (c) $\boldsymbol{F} = \langle yz, xz, xy \rangle$ along *any* curve between $(-1, -1, 2)$ to $(-2, 1, -1)$.
   answer: The field is conservative; use the potential. answer is 0
   (d) $\boldsymbol{F} = \left\langle 2xz + y^2z^2, 2xyz^2, x^2 + 2xy^2z \right\rangle$ along the curve parametrized by $\boldsymbol{R}(t) = \left\langle \cos(\pi t) + e^t, -\sin(\pi t) + \log(1 + t), t^{12} \right\rangle$ from $(2, 0, 0)$ to $(e - 1, \log 2, 1)$.
   answer: The field is conservative; use the potential. answer is $(e-1)\left[e - 1 + (\log 2)^2\right]$
2. Given a vector field $\boldsymbol{F}$, determine if it is conservative. If it is conservative, find ϕ such that $\boldsymbol{F} = \nabla\phi$. *e.g.*,
   (a) $\boldsymbol{F} = \left\langle 2xy, x^2 + z, y \right\rangle$.
   answer: $\phi = x^2y + zy$.
   (b) $\boldsymbol{F} = \langle xy, z, x \rangle$.
   answer: not conservative.
   (c) $\boldsymbol{F} = \left\langle x^2y + 1, \frac{1}{3}x^3 + 1, y \right\rangle$.
   answer: not conservative.
   (d) $\boldsymbol{F} = \langle -\sin x \cos y, \cos x \cos y, 1 \rangle$.
   answer: not conservative.
   (e) $\boldsymbol{F} = \frac{1}{x^2+y^2+z^2} \langle x, y, z \rangle$.
   answer: Doesn't $\phi = \frac{1}{2} \ln\left(x^2 + y^2 + z^2\right)$ work?
   (f) $\boldsymbol{F} = \frac{1}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle$.
   answer: Doesn't $\phi = -\frac{1}{\sqrt{x^2+y^2+z^2}}$ work?
3. Give two conditions equivalent to $\boldsymbol{F}$ being a conservative field.
   answer: (1) Every line integral is path-independent. (2) Every circulation is zero.
4. If $\boldsymbol{F}$ is conservative, what can we say about $\nabla \times \boldsymbol{F}$? Under what conditions does the converse hold? Why do we care if $\boldsymbol{F}$ is conservative? What does it allow us to do?
   answer: Curl is zero. Converse holds on a simply connected domain. It's easier to compute the line integral of a conservative field using the potential.
5. Suppose $\nabla\phi = \nabla\psi$. What more can we say about ϕ and ψ?
   answer: Differ by a constant.
6. Given some parametrized surface, S, find $\mathrm{d}\boldsymbol{S}$ and $\mathrm{d}S$ in terms of $\mathrm{d}u$ and $\mathrm{d}v$. *e.g.*,:
   (a) S is parametrized by $\boldsymbol{R}(u, v) = \left\langle v\cos u, v \sin u, v^2 \right\rangle$.
   answer: $\mathrm{d}\boldsymbol{S} = \left\langle -2v^2 \cos u, -2v^2 \sin u, v \right\rangle \mathrm{d}u\mathrm{d}v$.

(b) S is parametrized by $\boldsymbol{R}(u,v) = \langle (1+\cos u)\cos v, (1+\cos u)\sin v, \sin v\rangle$.
answer: $\mathrm{d}\boldsymbol{S} = \left\langle -\sin u \sin v \cos v, \sin u \cos^2 v, (1+\cos u)\sin u\right\rangle \mathrm{d}u\mathrm{d}v$.

7. Given some surface, S, find a parametrization $\boldsymbol{R}(u,v)$ of S. Find the appropriate limits for u, v. Find $\mathrm{d}\boldsymbol{S}$ and $\mathrm{d}S$ in terms of $\mathrm{d}u$ and $\mathrm{d}v$. Set up an integral for the surface area of S. *e.g.*,:
   (a) S is the part of the cone $x^2 = y^2 + z^2$ inside the sphere $(x-8)^2 + y^2 + z^2 = 25$.
   answer: As stated, this problem is bogus. Try instead:
   (b) S is the part of the cone $x^2 = y^2 + z^2$ inside the sphere $(x-8)^2 + y^2 + z^2 = 49$.
   answer: $\boldsymbol{R}(x,\theta) = \langle x, x\cos\theta, x\sin\theta\rangle, 4 - \sqrt{\frac{17}{2}} \leq x \leq 4 + \sqrt{\frac{17}{2}}$. And $\mathrm{d}\boldsymbol{S} = \langle x, -x\cos\theta, -x\sin\theta\rangle \mathrm{d}x\mathrm{d}\theta$.
   (c) S is the part of the paraboloid $z = x^2 + y^2$ above the xy plane and below the plane $x + y + z = 9$.
   answer: This problem is hard. I won't give the limits of z. $\boldsymbol{R}(z,\theta) = \langle \sqrt{z}\cos\theta, \sqrt{z}\sin\theta, z\rangle$. $\mathrm{d}\boldsymbol{S} = \left\langle \sqrt{z}\cos\theta, \sqrt{z}\sin\theta, -\frac{1}{2}\right\rangle \mathrm{d}z\mathrm{d}\theta$.

8. Given some surface, S, and a vector field $\boldsymbol{F}$ find the integral of the flux:

$$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}.$$

   e.g.,:
   (a) $\boldsymbol{F} = \langle 0, 0, \cos(xy + 2^z)\rangle$, S is the part of the cylinder $x^2 + y^2 = 1$ with $0 \leq z \leq 2$.
   answer: Obviously 0. The field is always orthogonal to the normal.
   (b) $\boldsymbol{F} = \langle x, y, -z\rangle$, S is the surface of the ellipsoid $x^2 + \frac{1}{4}y^2 + \frac{1}{9}z^2 = 9$.
   answer: By divergence theorem, this is the volume of the ellipsoid. I think this might be 216π. You can check this by using a change of variables to find the volume.
   (c) $\boldsymbol{F} = \left\langle x^2, y, z^3\right\rangle$, S is the surface of the cube bounded by $x = 0, x = 2, y = \pm 1, z = \pm 1$.
   answer: Directly or by divergence theorem, a boring calculation. 32.
   (d) $\boldsymbol{F} = \langle y + z, 2x + y, y\rangle$, S is the surface of the triangle with corners $(1,0,0)$, $(0,2,0)$, $(0,0,3)$ with normal pointing away from the origin.
   answer: The integral is something like

$$\int_0^1 \int_0^{2-2x} y - 6x + 9 \,\mathrm{d}y\mathrm{d}x.$$

   (e) $\boldsymbol{F} = \langle 2y, 2x, z\rangle$, S is the part of the cone $x^2 = y^2 + z^2$ inside the sphere $(x-8)^2 + y^2 + z^2 = 49$. Assume $\boldsymbol{n} \cdot \boldsymbol{i} < 0$.
   answer: I think this is $\frac{-113\pi}{3}\sqrt{\frac{17}{2}}$.
   (f) $\boldsymbol{F} = \left\langle 3, x^2, y\right\rangle$, S is the part of the paraboloid $z = x^2 + y^2$ above the xy plane and below the plane $x + y + z = 9$. Assume $\boldsymbol{n} \cdot \boldsymbol{k} > 0$.
   answer: The limits of z make this difficult. I should have made this a cone instead of a paraboloid.

You should be prepared to answer questions of this type directly. You should also be able to recognize when the Divergence Theorem applies, and be able to apply it.

9. State the Divergence Theorem. State Stokes' Theorem.
   answer: You need to know these cold.
10. Compute a "volume" integral *i.e.*, compute

$$\iint_R f(x,y)\mathrm{d}A \quad \text{or} \quad \iiint_R f(x,y,z)\mathrm{d}V$$

by converting to (and evaluating) an iterated integral:

$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\mathrm{d}y\mathrm{d}x \quad \text{or} \quad \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

   (a) $f(x,y) = 2xy$, R is bounded by $y = 0, x = 2, y = x^2$.
   answer: This is Shenk's §15.1 example 3. Answer is 32/3.
   (b) $f(x,y) = y^2 \cos x$, R is bounded by $x = y^3, y = -1, y = 1, x = 3$.
   answer: This is Shenk's §15.1 worked problem #2. Answer is cos(1).
   (c) $f(x,y,z) = \sqrt{x^2+z^2}$, R is bounded by $y = x^2 + z^2, y = 4$.
   answer: I think this is $\frac{256\pi}{45}\sqrt{2}$.
   (d) $f(x,y,z) = x + y + z$, R is bounded by $x = y^2, x = z, z = 0, z = 1$.
   answer: I think there was an error here. the last equality should likely be $x = 1$. In this case the answer is 6/7.

You should be able to effortlessly convert to polar, cylindrical, or spherical coordinates and correctly add the appropriate integrating factors.

11. You should be able to compute a "volume" integral by making a change of variables. You should be able to compute the Jacobian of the transformation. You should be able to compute the Jacobian for an "implicit" transformation (one in which u, v are given in terms of x, y and not the other way around). You should be able to figure out what the transformed region, $\tilde{R}$ looks like, and how to figure out its extents.
   answer: Some of these are from Shenk's Handout, in the "Worked Examples Section."
   (a) $f(x,y) = \sin\left(x^2 + 2xy + y^2\right)$, R is the triangle with corners (0,0),(1,0),(0,1) using the change $u = x + y, v = x - y$.
   answer: This is Shenk's §15.6 worked problem #2. Answer is $-\frac{1}{2}\cos(1) + \frac{1}{2}$.
   (b) $f(x,y) = e^{xy}$, R is bounded by $xy = 1, xy = 4, y = 1, y = 3$ using the change $x = u/v, y = v$.
   answer: This is Shenk's §15.6 worked problem #3. Answer is $\left(e^4 - e\right)\ln 3$.
   (c) $f(x,y,z) = \frac{2y}{x+2y+4z}$, R is bounded by $1 \leq x + 2y + 4z \leq 4$, $0 \leq y - 2z \leq 1$, $0 \leq y + 2z \leq 2$, using an affine transform.
   answer: This is Shenk's §15.6 worked problem #4. Answer is $\frac{3}{4}\ln 4$.
   (d) $f(x,y) = (x-y)^3$, R is bounded by $y = x, y = 3x, x = 3/2$, using the transform $u = x - y, v = 3x - y$.
   answer: I think the answer is $-\frac{243}{40}$.
   (e) $f(x,y,z) = \frac{1}{\sqrt{x^2+\frac{1}{9}y^2+\frac{1}{16}z^2}}$, with R the ellipsoid $x^2 + \frac{1}{9}y^2 + \frac{1}{16}z^2 \leq 25$.
   answer: I think the answer is $(12)(50\pi) = 600\pi$.