Final Exam Preparation

The following formulæ will be provided on your exam:

$$\mathrm{d}\boldsymbol{R} = \boldsymbol{R}'(t)\mathrm{d}t \quad \mathrm{d}\boldsymbol{S} = \left(\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}x} \times \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}y}\right)\mathrm{d}x\mathrm{d}y \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

Everything else must be committed to memory.

What follows are questions similar to your homework assignments. You should be prepared to answer at least all these questions. In no way should you consider this list exhaustive.

Beware that I have not yet tried to answer some of these questions, and they may be challenging or impossible. If you know how to answer the question in general and get stuck on algebra or something does not seem right, the problem may be bogus. These inconsistencies may be corrected if and when I get around to answering these questions.

1. Given scalar field $\phi(x,y,z)$, find the gradient $\nabla\phi$. *e.g.*,:
   (a) $\phi(x,y,z) = x^2 + y^2 + z^2$.
   (b) $\phi(x,y,z) = x^2 + y^2 - z^2$.
   (c) $\phi(x,y,z) = \frac{1}{4}x^2 + \frac{1}{25}y^2 + z^2$.
   (d) $\phi(x,y,z) = xyz + x^2$.
   (e) $\phi(x,y,z) = \cos(xy) + \sin(yz)$.
2. Given scalar field $\phi(x,y,z)$, and direction $\boldsymbol{u}$, find the directional derivative $\mathrm{D}_{\boldsymbol{u}}\,\phi$ both as a scalar field and evaluated at a point. *e.g.*,:
   (a) $\phi(x,y,z) = x^2 + y^2 + z^2$, $\boldsymbol{u} = \langle 0,1,0\rangle$.
   (b) $\phi(x,y,z) = x^2 + y^2 - z^2$, $\boldsymbol{u} = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle$.
   (c) $\phi(x,y,z) = \frac{1}{4}x^2 + \frac{1}{25}y^2 + z^2$, $\boldsymbol{u} = \frac{1}{\sqrt{2}}\langle 0,1,1\rangle$.
   (d) $\phi(x,y,z) = xyz + x^2$, $\boldsymbol{u} = \langle 1,0,0\rangle$ at the point $(2,1,0)$.
3. Given scalar field $\phi(x,y,z)$, describe the level sets ("isotimic surfaces") of the field. Find a normal vector to the level sets, and describe the flow field of ϕ. *e.g.*,:
   (a) $\phi(x,y,z) = x^2 + y^2 + z^2$.
   (b) $\phi(x,y,z) = x^2 + y^2 - z^2$.
   (c) $\phi(x,y,z) = \frac{1}{4}x^2 + \frac{1}{25}y^2 + z^2$.
4. For vector field $\boldsymbol{F}(x,y,z)$, find the divergence, $\nabla\cdot\boldsymbol{F}$, and the curl $\nabla\times\boldsymbol{F}$. *e.g.*,:
   (a) $\boldsymbol{F}(x,y,z) = \langle x,y,z\rangle$.
   (b) $\boldsymbol{F}(x,y,z) = \langle -y,x,0\rangle$.
   (c) $\boldsymbol{F}(x,y,z) = \langle y,z,x\rangle$.
   (d) $\boldsymbol{F}(x,y,z) = \langle x,y,z\rangle + \langle -y,x,0\rangle$.
   (e) $\boldsymbol{F}(x,y,z) = \langle x^2+y^2, 3xy^2, 0\rangle$.
   (f) $\boldsymbol{F}(x,y,z) = \langle y - \sin x, \cos x, 0\rangle$.
   (g) $\boldsymbol{F}(x,y,z) = \langle y - \sin x, \cos x, 0\rangle + \nabla\phi$, where ϕ is harmonic, *i.e.*, $\nabla^2\phi = 0$.
   (h) $\boldsymbol{F}(x,y,z) = \left(x^2+y^2+z^2\right)\langle x,y,z\rangle$.
   (i) $\boldsymbol{F}(x,y,z) = \left(x^2+y^2-z^2\right)\langle y,z,x\rangle$.
5. Define what makes a vector field conservative. Give some equivalent conditions to a vector field being conservative.

6. Given a vector field $\boldsymbol{F}(x,y,z)$, determine if it is conservative. If it is, find a potential for the field. *e.g.*,:
   (a) $\boldsymbol{F}(x,y,z) = \langle x,y,z\rangle$.
   (b) $\boldsymbol{F}(x,y,z) = \langle -y,x,0\rangle$.
   (c) $\boldsymbol{F}(x,y,z) = \langle y,z,x\rangle$.
   (d) $\boldsymbol{F}(x,y,z) = \langle x,y,z\rangle + \langle -y,x,0\rangle$.
   (e) $\boldsymbol{F}(x,y,z) = \left(x^2+y^2+z^2\right)^k \langle x,y,z\rangle$.
   (f) $\boldsymbol{F}(x,y,z) = \left\langle yz+z^2, xz, xy+2xz\right\rangle$.
   (g) $\boldsymbol{F}(x,y,z) = \cos\left(\frac{xy}{z}\right)\left\langle \frac{y}{z}, \frac{x}{z}, \frac{-xy}{z^2}\right\rangle$.
   (h) $\boldsymbol{F}(x,y,z) = \langle yz+2x, xz, xy+x\rangle$.
7. Given a curve C, parametrize the curve by $\boldsymbol{R}(t)$; or, given a surface S, parametrize the surface by $\boldsymbol{R}(u,v)$. In either case, find the appropriate limits for the independent variables. For the curve find $\mathrm{d}\boldsymbol{R}$ in terms of $\mathrm{d}t$; for the surface, find find $\mathrm{d}\boldsymbol{S}$ and $\mathrm{d}S$ in terms of $\mathrm{d}u$ and $\mathrm{d}v$. *e.g.*,:
   (a) C is the circle $x^2+y^2=4$.
   (b) C is the ellipse $(x/a)^2+(y/b)^2=1$.
   (c) C is the semicircle $x^2+y^2=1$, with $x>y$.
   (d) S is the ellipsoid $(x/a)^2+(y/b)^2+(z/c)^2=1$.
   (e) S is the part of the paraboloid $z=x^2+y^2$ with $0\le z\le 4$. Let S be oriented such that $\boldsymbol{n}\cdot\boldsymbol{k}<0$.
   (f) S is the part of the cone $z^2=x^2+y^2$ above the xy plane and below the plane $5x+2y-3z=-3$. Let S be oriented such that $\boldsymbol{n}\cdot\boldsymbol{k}>0$.
   (g) S is the part of the cylinder $y^2+z^2=4$ with $0\le x$, and $x-y-2z\le 8$. Let S be oriented such that $\boldsymbol{n}$ points away from the x axis.
   (h) S is the part of the cylinder $x^2+y^2=1$ inside the sphere $x^2+y^2+(z-4)^2=16$. Let S be oriented such that $\boldsymbol{n}$ points away from the z axis.
8. Given a vector field $\boldsymbol{F}$, and some curve C, determine the "integral of the tangential component" of $\boldsymbol{F}$ along C. You should be prepared to do this "the long way" when $\boldsymbol{F}$ is not conservative, and by the shortcut when it is. It may be possible and adviseable to use Stokes' Theorem (or Green's Theorem) when C is closed and $\boldsymbol{F}$ is not conservative. *e.g.*,:
   (a) $\boldsymbol{F} = \langle 1,2y\rangle$, along $x^2+y^2=4$. (this is a closed curve)
   (b) $\boldsymbol{F} = \langle -y,x,2\rangle$ along the straight line from $(0,0,0)$ to $(2,1,3)$.
   (c) $\boldsymbol{F} = \langle y,x+z,y\rangle$ along *any* curve between $(1,1,2)$ to $(2,-1,1)$.
   (d) $\boldsymbol{F} = \langle -y,x,4z\rangle$ along the part of the helix parametrized by $\langle 4\cos t, 4\sin t, 4t\rangle$ for $0\le t\le 1$.
   (e) $\boldsymbol{F} = \left\langle 3x^2+2y, -x-3\cos y, 0\right\rangle$ with C the square with corners $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(0,1,0)$, oriented from $(0,0,0)$ to $(1,0,0)$.
   (f) $\boldsymbol{F} = \left\langle x^2+y^2, 3xy^2, 0\right\rangle$ with C the closed curve consisting of the straight line from $(2,-4,0)$ to $(2,4,0)$, then back to $(2,-4,0)$ along the curve $y^2=8x$.
   (g) $\boldsymbol{F} = \langle y-\sin x, \cos x, 0\rangle$ with C the triangle with corners $(0,0,0)$, $(\pi/2,0,0)$, $(\pi/2,1,0)$ oriented counterclockwise in the xy plane.
   (h) $\boldsymbol{F} = e^{xyz}\langle yz,xz,xy\rangle$ along the curve parametrized by $\left\langle (1-t)^5, \cos(\pi t), t\right\rangle$ from $(1,1,0)$ to $(0,-1,1)$.

(i) $\boldsymbol{F} = \left\langle 3x^2 + 6y, -14yz, 20xz^2 \right\rangle$ along the curve parametrized by $\left\langle t, t^2, t^3 \right\rangle$ from $(0, 0, 0)$ to $(1, 1, 1)$.

9. Given some surface, S, and a vector field $\boldsymbol{F}$ find the integral of the flux:

$$\iint\limits_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}.$$

You should be prepared to answer questions of this type directly when necessary; you should also be able to recognize when the Divergence Theorem or Stokes' Theorem applies, and be able to apply them. *e.g.*,:

(a) $\boldsymbol{F} = \left\langle e^{2xyz}, 0, 0 \right\rangle$, S is the part of the cylinder $y^2 + z^2 = 1$ with $0 \leq x \leq 2$.
(b) $\boldsymbol{F} = \left\langle \frac{y}{b^2}, \frac{-x}{a^2}, 1 \right\rangle$, S is the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$.
(c) $\boldsymbol{F} = \langle x + z, x + y, \cos(x) - z \rangle$, S is the surface of some set D with volume V.
(d) $\boldsymbol{F} = \langle y - z, z - 2x, y \rangle$, S is the surface of the triangle with corners $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 3)$ with normal pointing away from the origin.
(e) $\boldsymbol{F} = \langle 2y, 2x, 2z \rangle$, S is the part of the cylinder $x^2 + y^2 = 1$ inside the sphere $x^2 + y^2 + (z - 4)^2 = 16$. Let S be oriented such that $\boldsymbol{n}$ points away from the z axis.
(f) $\boldsymbol{F} = \left\langle x^2, y^2, z^3 \right\rangle$, S is the surface of the cube bounded by $x = \pm 1, y = \pm 1, z = \pm 1$.
(g) $\boldsymbol{F} = \langle y, x + z, x \rangle$, S is the graph of $f(x, y) = x^2 + y$ over the region in the xy plane bounded by $0 \leq x \leq 1, 0 \leq y \leq 1 + x^2$, oriented such that $\boldsymbol{n} \cdot \boldsymbol{k} > 0$.
(h) $\boldsymbol{F} = \nabla \times \left\langle x^2 + y - 4, 3xy, 2xz + z^2 \right\rangle$, S is the surface of the hemisphere $x^2 + y^2 + z^2 = 16$ above the xy plane.
(i) $\boldsymbol{F} = \langle y, -x, z \rangle$, S is the part of the parabola $z = x^2 + y^2$ below the plane $z = x$, oriented such that $\boldsymbol{n} \cdot \boldsymbol{k} < 0$. (May be difficult)

10. State the Divergence Theorem. State Stokes' Theorem.

11. Compute a double or triple integral *i.e.*, compute

$$\iint_R f(x, y)\mathrm{d}A \quad \text{or} \quad \iiint_R f(x, y, z)\mathrm{d}V$$

by converting to (and evaluating) an iterated integral:

$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x, y)\mathrm{d}y\mathrm{d}x \quad \text{or} \quad \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

If necessary convert to polar, cylindrical, or spherical coordinates and correctly add the appropriate integrating factors.

(a) $f(x, y) = x + y$, R is bounded by $x = 0, y = x^2 - 1, y = 5 - x^2$.
(b) $f(x, y) = y \cos x$, R is bounded by $x = y^2, y = -1, y = 1, x = 3$.
(c) $f(x, y, z) = 1$, R is bounded by $x^2 + y^2 + z^2 = 4$, and $z = 1$.
(d) $f(x, y, z) = z^2$, R is the region in the sphere $x^2 + y^2 + z^2 = 4$, and outside the cylinder $x^2 + y^2 = 1$.
(e) $f(x, y, z) = \sqrt{x^2 + y^2}$, R is the region $x^2 + y^2 \leq 4$, $4 - x^2 - y^2 \leq z \leq 4$.
(f) $f(x, y, z) = 1$, R is the region $1 \leq x^2 + y^2 + z^2 \leq 4$, with $z > 0$.

12. Given a function f and a region R, compute the integral

$$\iint\limits_R f \mathrm{d}A \quad \text{or} \quad \iiint\limits_R f \mathrm{d}V$$

by making a given change of variables and inserting the (determinant of the) Jacobian of the transformation. You should be able to compute the Jacobian for an "implicit" transformation (one in which u, v are given in terms of x, y and not the other way around). You should be able to figure out what the transformed region, $\tilde{R}$ looks like, and how to figure out its extents.

   (a) $f(x, y, z) = x^2 - y^2$, where R is the region bounded by $y \leq x \leq y + 1$, and $-y \leq x \leq 1 - x$ by making the change $u = x + y, v = x - y$.

   (b) $f(x, y, z) = \left[(x/a)^2 + (y/b)^2 + (z/c)^2\right]^k$, for some number k. Let R be the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1$. Make the transformation to u, v, w such that $\tilde{R}$ is a sphere.

   (c) $f(x, y, z) = \frac{3x}{2x-y+z}$, R is bounded by $0 \leq x - z \leq 1$, $0 \leq 2x + z \leq 1$, and $2 \leq 2x - y + z \leq 6$, by using an affine transform.

   (d) $f(x, y, z) = x^2 + y^2$, R is a torus, $\tilde{R}$ is the rectangle $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq 2\pi$, and $0 \leq r \leq a$, under the transform:

$$\begin{aligned} x &= A\cos\phi + r\cos\phi\cos\theta \\ y &= A\sin\phi + r\sin\phi\cos\theta \\ z &= r\sin\theta, \end{aligned}$$

   with $a \leq A$.

   (e) $f(x, y, z) = \frac{2y}{x+2y+4z}$, R is bounded by $1 \leq x + 2y + 4z \leq 4$, $0 \leq y - 2z \leq 1$, $0 \leq y + 2z \leq 2$, using an affine transform.

   (f) $f(x, y) = e^{xy}$, R is bounded by $xy = 1, xy = 4, y = 1, y = 3$ using the change $x = u/v, y = v$. (A repeat from ex2 prep sheet, and stolen from Shenk's handout)

   (g) $f(x, y) = (x - y)^3$, R is bounded by $y = x, y = 3x, x = 3/2$, using the transform $u = x - y, v = 3x - y$. (A repeat from ex2 prep sheet)

13. Given some region in the plane, R, bounded by a curve C, use Green's Theorem in the plane to evaluate A, the area of R by computing a line integral of some field along C. Note that the field is likely not going to be conservative, unless R has zero area. *e.g.*,:

   (a) Let C be parametrized by $\boldsymbol{R}(t) = \langle\cos 3t\cos t, \cos 3t\sin t, 0\rangle$, for $-\pi/6 \leq t \leq \pi/6$.

   (b) Let C be parametrized by $\boldsymbol{R}(t) = \langle\cos^2 t, \cos t\sin t, 0\rangle$, for $-\pi/2 \leq t \leq \pi/2$. (you should be able to solve this without any calculus, I believe)

   (c) Let C be parametrized by $\boldsymbol{R}(t) = \langle\cos^3 t, \cos^2 t\sin t, 0\rangle$, for $-\pi/2 \leq t \leq \pi/2$.

   (d) Let C be one loop of the lemniscate parametrized by $\boldsymbol{R}(t) = \left\langle\sqrt{\cos 2t}\cos t, \sqrt{\cos 2t}\sin t, 0\right\rangle$ for $-\pi/4 \leq t \leq \pi/4$.

14. Understand that the Laplacian is $\nabla^2 = \nabla \cdot \nabla$. Appreciate how Green's first formula is derived from the Divergence Theorem applied to the field $\phi\nabla\psi$.