Final Exam Preparation

The following formulæ will be provided on your exam:

$$\mathrm{d}\boldsymbol{R} = \boldsymbol{R}'(t)\mathrm{d}t \quad \mathrm{d}\boldsymbol{S} = \left(\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}x} \times \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}y}\right)\mathrm{d}x\mathrm{d}y \quad \frac{\partial(x,y)}{\partial\,(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Everything else must be committed to memory.

What follows are questions similar to your homework assignments. You should be prepared to answer at least all these questions. In no way should you consider this list exhaustive.

1. Given scalar field $\phi(x,y,z)$, find the gradient $\nabla\phi$. *e.g.*,:
   (a) $\phi(x,y,z) = x^2 + y^2 + z^2$.
   answer: $\langle 2x, 2y, 2z\rangle$
   (b) $\phi(x,y,z) = x^2 + y^2 - z^2$.
   answer: $\langle 2x, 2y, -2z\rangle$
   (c) $\phi(x,y,z) = \frac{1}{4}x^2 + \frac{1}{25}y^2 + z^2$.
   answer: $\langle 2x/4, 2y/25, 2z\rangle$
   (d) $\phi(x,y,z) = xyz + x^2$.
   answer: $\langle yz + 2x, xz, xy\rangle$
   (e) $\phi(x,y,z) = \cos(xy) + \sin(yz)$.
   answer: $\langle -y\sin(xy), -x\sin(xy) + z\cos(y,z), y\cos(yz)\rangle$
2. Given scalar field $\phi(x,y,z)$, and direction $\boldsymbol{u}$, find the directional derivative $\mathrm{D}_{\boldsymbol{u}}\,\phi$ both as a scalar field and evaluated at a point. *e.g.*,:
   (a) $\phi(x,y,z) = x^2 + y^2 + z^2$, $\boldsymbol{u} = \langle 0,1,0\rangle$.
   answer: $\mathrm{D}_{\boldsymbol{u}}\,\phi = 2y$
   (b) $\phi(x,y,z) = x^2 + y^2 - z^2$, $\boldsymbol{u} = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle$.
   answer: $\mathrm{D}_{\boldsymbol{u}}\,\phi = \frac{2}{\sqrt{3}}(x + y - z)$
   (c) $\phi(x,y,z) = \frac{1}{4}x^2 + \frac{1}{25}y^2 + z^2$, $\boldsymbol{u} = \frac{1}{\sqrt{2}}\langle 0,1,1\rangle$.
   answer: $\mathrm{D}_{\boldsymbol{u}}\,\phi = \frac{2}{\sqrt{2}}\left(\frac{1}{25}y + z\right)$
   (d) $\phi(x,y,z) = xyz + x^2$, $\boldsymbol{u} = \langle 1,0,0\rangle$ at the point $(2,1,0)$.
   answer: $\mathrm{D}_{\boldsymbol{u}}\,\phi\,(2,1,0) = 4$
3. Given scalar field $\phi(x,y,z)$, describe the level sets ("isotimic surfaces") of the field. Find a normal vector to the level sets, and describe the flow field of ϕ. *e.g.*,:
   (a) $\phi(x,y,z) = x^2 + y^2 + z^2$.
   answer: each level set is a sphere; the position vector $\langle x,y,z\rangle$ is normal to the level set; the flow lines are rays extending from the origin to infinity.
   (b) $\phi(x,y,z) = x^2 + y^2 - z^2$.
   answer: each level set is a cone; the vector $\langle x,y,-z\rangle$ is normal to the level set; the flow lines are complicated.
   (c) $\phi(x,y,z) = \frac{1}{4}x^2 + \frac{1}{25}y^2 + z^2$.
   answer: each level set is an ellipsoid; the vector $\left\langle \frac{1}{4}x, \frac{1}{25}y, z\right\rangle$ is normal to the level set; the flow lines are complicated.
4. For vector field $\boldsymbol{F}(x,y,z)$, find the divergence, $\nabla\cdot\boldsymbol{F}$, and the curl $\nabla\times\boldsymbol{F}$. *e.g.*,:
   (a) $\boldsymbol{F}(x,y,z) = \langle x,y,z\rangle$.
   answer: $\nabla\cdot\boldsymbol{F} = 3$, $\nabla\times\boldsymbol{F} = \boldsymbol{0}$.

(b) $\boldsymbol{F}(x,y,z) = \langle -y, x, 0\rangle$.
answer: $\nabla \cdot \boldsymbol{F} = 0$, $\nabla \times \boldsymbol{F} = \langle 0,0,2\rangle$.

(c) $\boldsymbol{F}(x,y,z) = \langle y, z, x\rangle$.
answer: $\nabla \cdot \boldsymbol{F} = 0$, $\nabla \times \boldsymbol{F} = \langle -1,-1,-1\rangle$.

(d) $\boldsymbol{F}(x,y,z) = \langle x, y, z\rangle + \langle -y, x, 0\rangle$.
answer: $\nabla \cdot \boldsymbol{F} = 3$, $\nabla \times \boldsymbol{F} = \langle 0,0,2\rangle$.

(e) $\boldsymbol{F}(x,y,z) = \left\langle x^2+y^2, 3xy^2, 0\right\rangle$.
answer: $\nabla \cdot \boldsymbol{F} = 2x + 6xy$, $\nabla \times \boldsymbol{F} = \left\langle 0,0,3y^2-2y\right\rangle$.

(f) $\boldsymbol{F}(x,y,z) = \langle y - \sin x, \cos x, 0\rangle$.
answer: $\nabla \cdot \boldsymbol{F} = -\cos x$, $\nabla \times \boldsymbol{F} = \langle 0,0,-\sin x - 1\rangle$.

(g) $\boldsymbol{F}(x,y,z) = \langle y - \sin x, \cos x, 0\rangle + \nabla\phi$, where ϕ is harmonic, *i.e.*, $\nabla^2\phi = 0$.
answer: $\nabla \cdot \boldsymbol{F} = -\cos x$, $\nabla \times \boldsymbol{F} = \langle 0,0,-\sin x - 1\rangle$.

(h) $\boldsymbol{F}(x,y,z) = \left(x^2+y^2+z^2\right)\langle x, y, z\rangle$.
answer: $\nabla \cdot \boldsymbol{F} = 5\left(x^2+y^2+z^2\right)$, $\nabla \times \boldsymbol{F} = \mathbf{0}$.

(i) $\boldsymbol{F}(x,y,z) = \left(x^2+y^2-z^2\right)\langle y, z, x\rangle$.
answer: $\nabla \cdot \boldsymbol{F} = 2xy + 2yz - 2zx$,
$\nabla \times \boldsymbol{F} = \left\langle 2yz + 3z^2 - x^2 - y^2, -2zy - 3x^2 - y^2 + z^2, 2xz - 3y^2 - x^2 + z^2\right\rangle$.

5. Define what makes a vector field conservative. Give some equivalent conditions to a vector field being conservative.
answer: A conservative field is one which is the gradient of some potential function. Three equivalent conditions are:
   (a) Every line integral of the field is path independent.
   (b) Every circulation of the field is equal to zero.
   (c) (When the domain is simply connected) the curl of the field is zero. (the field is "irrotational.")

6. Given a vector field $\boldsymbol{F}(x,y,z)$, determine if it is conservative. If it is, find a potential for the field. *e.g.*,:

(a) $\boldsymbol{F}(x,y,z) = \langle x, y, z\rangle$.
answer: Conservative, $\phi = \frac{1}{2}\left(x^2+y^2+z^2\right)$

(b) $\boldsymbol{F}(x,y,z) = \langle -y, x, 0\rangle$.
answer: Not conservative, it violates Clairaut's Theorem: $\phi_{xy} = -1 \neq 1 = \phi_{yx}$.

(c) $\boldsymbol{F}(x,y,z) = \langle y, z, x\rangle$.
answer: Not conservative as it violates Clairaut's Theorem: $\phi_{xy} = 1 \neq 0 = \phi_{yx}$.

(d) $\boldsymbol{F}(x,y,z) = \langle x, y, z\rangle + \langle -y, x, 0\rangle$.
answer: Not conservative; the sum or difference of conservative fields is conservative. Thus if this field were conservative when we subtract off the conservative field $\langle x, y, z\rangle$ we would be left with a conservative field; but we saw previously that $\langle -y, x, 0\rangle$ is *not* conservative.
Or you can directly find that this function is not conservative, as it would violate Clairaut's Theorem: $\phi_{xy} = -1 \neq 1 = \phi_{yx}$.

(e) $\boldsymbol{F}(x,y,z) = \left(x^2+y^2+z^2\right)^k\langle x, y, z\rangle$.
answer: Conservative, $\phi = \frac{1}{2(k+1)}\left(x^2+y^2+z^2\right)^{k+1}$

(f) $\boldsymbol{F}(x,y,z) = \left\langle yz + z^2, xz, xy + 2xz\right\rangle$.
answer: Conservative, $\phi = xyz + xz^2$

(g) $\boldsymbol{F}(x,y,z) = \cos\left(\frac{xy}{z}\right)\left\langle \frac{y}{z}, \frac{x}{z}, \frac{-xy}{z^2}\right\rangle.$
answer: Conservative, $\phi = \cos\left(\frac{xy}{z}\right)$

(h) $\boldsymbol{F}(x,y,z) = \langle yz + 2x, xz, xy + x\rangle.$
answer: Not Conservative as $\phi_{zx} \neq \phi_{xz}$.

7. Given a curve C, parametrize the curve by $\boldsymbol{R}(t)$; or, given a surface S, parametrize the surface by $\boldsymbol{R}(u,v)$. In either case, find the appropriate limits for the independent variables. For the curve find $\mathrm{d}\boldsymbol{R}$ in terms of $\mathrm{d}t$; for the surface, find $\mathrm{d}\boldsymbol{S}$ and $\mathrm{d}S$ in terms of $\mathrm{d}u$ and $\mathrm{d}v$. *e.g.*,:

(a) C is the circle $x^2 + y^2 = 4$.
answer: $\boldsymbol{R}(t) = \langle 2\cos t, 2\sin t\rangle, 0 \le t \le 2\pi$. $\boldsymbol{R}'(t) = \langle -2\sin t, 2\cos t\rangle\,\mathrm{d}t$.

(b) C is the ellipse $(x/a)^2 + (y/b)^2 = 1$.
answer: $\boldsymbol{R}(t) = \langle a\cos t, b\sin t\rangle, 0 \le t \le 2\pi$. $\boldsymbol{R}'(t) = \langle -a\sin t, b\cos t\rangle\,\mathrm{d}t$.

(c) C is the semicircle $x^2 + y^2 = 1$, with $x > y$.
answer: $\boldsymbol{R}(t) = \langle \cos t, \sin t\rangle, -\frac{3\pi}{4} \le t \le \frac{\pi}{4}$. $\boldsymbol{R}'(t) = \langle -\sin t, \cos t\rangle\,\mathrm{d}t$.

(d) S is the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$.
answer: $\boldsymbol{R}(u,v) = \langle a\sin u\cos v, b\sin u\sin v, c\cos u\rangle, 0 \le u \le \pi, 0 \le v \le 2\pi$.
$\mathrm{d}\boldsymbol{S} = \left\langle bc\sin^2 u\cos v, ac\sin^2 u\sin v, ab\cos u\sin u\right\rangle \mathrm{d}u\mathrm{d}v$,
$\mathrm{d}S = abc\,|\sin u|\sqrt{\left(\frac{\sin u\cos v}{a}\right)^2 + \left(\frac{\sin u\sin v}{b}\right)^2 + \left(\frac{\cos u}{c}\right)^2}\mathrm{d}u\mathrm{d}v$

(e) S is the part of the paraboloid $z = x^2 + y^2$ with $0 \le z \le 4$. Let S be oriented such that $\boldsymbol{n}\cdot\boldsymbol{k} < 0$.
answer: $\boldsymbol{R}(x,y) = \left\langle x, y, x^2 + y^2\right\rangle, 0 \le x^2 + y^2 \le 4$.
$\mathrm{d}\boldsymbol{S} = \langle 2x, 2y, -1\rangle\,\mathrm{d}x\mathrm{d}y$, $\mathrm{d}S = \sqrt{1 + 4x^2 + 4y^2}\mathrm{d}x\mathrm{d}y$.
Alternatively you could try $\boldsymbol{R}(r,\theta) = \left\langle r\cos\theta, r\sin\theta, r^2\right\rangle, 0 \le r \le 2, 0 \le \theta \le 2\pi$.
$\mathrm{d}\boldsymbol{S} = \langle 2r\cos\theta, 2r\sin\theta, -1\rangle\, r\mathrm{d}r\mathrm{d}\theta$, $\mathrm{d}S = \sqrt{1 + 4r^2}r\mathrm{d}r\mathrm{d}\theta$.

(f) S is the part of the cone $z^2 = x^2 + y^2$ above the xy plane and below the plane $5x + 2y - 3z = -3$. Let S be oriented such that $\boldsymbol{n}\cdot\boldsymbol{k} > 0$.
answer: $\boldsymbol{R}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle, 0 \le \theta \le 2\pi, 0 \le r \le \frac{3}{3 - 5\cos\theta - 2\sin\theta}$.
$\mathrm{d}\boldsymbol{S} = \langle -r\cos\theta, -r\sin\theta, r\rangle\,\mathrm{d}r\mathrm{d}\theta$, $\mathrm{d}S = \sqrt{2}r\mathrm{d}r\mathrm{d}\theta$.

(g) S is the part of the cylinder $y^2 + z^2 = 4$ with $0 \le x$, and $x - y - 2z \le 8$. Let S be oriented such that $\boldsymbol{n}$ points away from the x axis.
answer: $\boldsymbol{R}(x,\theta) = \langle x, 2\cos\theta, 2\sin\theta\rangle, 0 \le \theta \le 2\pi, 0 \le x \le 8 + 2\cos\theta + 4\sin\theta$.
$\mathrm{d}\boldsymbol{S} = \langle 0, 2\cos\theta, 2\sin\theta\rangle\,\mathrm{d}x\mathrm{d}\theta$, $\mathrm{d}S = 2\mathrm{d}x\mathrm{d}\theta$.

(h) S is the part of the cylinder $x^2 + y^2 = 1$ inside the sphere $x^2 + y^2 + (z-4)^2 = 16$. Let S be oriented such that $\boldsymbol{n}$ points away from the z axis.
answer: $\boldsymbol{R}(z,\theta) = \langle \cos\theta, \sin\theta, z\rangle, 0 \le \theta \le 2\pi, 4 - \sqrt{15} \le z \le 4 + \sqrt{15}$.
$\mathrm{d}\boldsymbol{S} = \langle \cos\theta, \sin\theta, 0\rangle\,\mathrm{d}z\mathrm{d}\theta$, $\mathrm{d}S = \mathrm{d}z\mathrm{d}\theta$.

8. Given a vector field $\boldsymbol{F}$, and some curve C, determine the “integral of the tangential component” of $\boldsymbol{F}$ along C. You should be prepared to do this “the long way” when $\boldsymbol{F}$ is not conservative, and by the shortcut when it is. It may be possible and adviseable to use Stokes’ Theorem (or Green’s Theorem) when C is closed and $\boldsymbol{F}$ is not conservative. *e.g.*,:

(a) $\boldsymbol{F} = \langle 1, 2y\rangle$, along $x^2 + y^2 = 4$. (this is a closed curve)
answer: 0 You can do this quickly by Green’s Theorem, as $(\nabla \times \langle 1, 2y, 0\rangle)\cdot\boldsymbol{k} = 0$. Or do it the long way.

(b) $\boldsymbol{F} = \langle -y, x, 2\rangle$ along the straight line from $(0,0,0)$ to $(2,1,3)$.
answer: 6, the long way; the field is not conservative, the curve not closed.

(c) $\boldsymbol{F} = \langle y, x+z, y\rangle$ along *any* curve between $(1,1,2)$ to $(2,-1,1)$.
answer: The words "any curve" are a code; they mean "this field is conservative." $\boldsymbol{F} = \nabla(xy+zy)$. The answer is -6.

(d) $\boldsymbol{F} = \langle -y, x, 4z\rangle$ along the part of the helix parametrized by $\langle 4\cos t, 4\sin t, 4t\rangle$ for $0 \le t \le 1$.
answer: 80, the long way; the field is not conservative, the curve not closed.

(e) $\boldsymbol{F} = \left\langle 3x^2+2y, -x-3\cos y, 0\right\rangle$ with C the square with corners $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(0,1,0)$, oriented from $(0,0,0)$ to $(1,0,0)$.
answer: The field is not conservative, but the curve is closed. The curve is in the xy plane, so we can apply Green's Theorem; $(\nabla \times \boldsymbol{F})\cdot\boldsymbol{k} = -3$, so the answer is -3 times the area enclosed by C. That is -3.

(f) $\boldsymbol{F} = \left\langle x^2+y^2, 3xy^2, 0\right\rangle$ with C the closed curve consisting of the straight line from $(2,-4,0)$ to $(2,4,0)$, then back to $(2,-4,0)$ along the curve $y^2 = 8x$.
answer: I must have miscopied the answer before. I have worked this one out, and I think you should get $\frac{512}{5}$. I did this by Stokes' Theorem, integrating $3y^2 - 2y$ over the region enclosed by the curve.

(g) $\boldsymbol{F} = \langle y - \sin x, \cos x, 0\rangle$ with C the triangle with corners $(0,0,0)$, $(\pi/2,0,0)$, $(\pi/2,1,0)$ oriented counterclockwise in the xy plane.
answer: should be $-\pi/4 - 2/\pi$.

(h) $\boldsymbol{F} = e^{xyz}\langle yz, xz, xy\rangle$ along the curve parametrized by $\left\langle (1-t)^5, \cos(\pi t), t\right\rangle$ from $(1,1,0)$ to $(0,-1,1)$.
answer: When the curve is parametrized by something really ugly, you usually have to rely on the field being conservative. In this case $\boldsymbol{F} = \nabla\left(e^{xyz}\right)$. The answer is 0.

(i) $\boldsymbol{F} = \left\langle 3x^2+6y, -14yz, 20xz^2\right\rangle$ along the curve parametrized by $\left\langle t, t^2, t^3\right\rangle$ from $(0,0,0)$ to $(1,1,1)$.
answer: should be 5.

9. Given some surface, S, and a vector field $\boldsymbol{F}$ find the integral of the flux:

$$\iint_S \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{S}.$$

You should be prepared to answer questions of this type directly when necessary; you should also be able to recognize when the Divergence Theorem or Stokes' Theorem applies, and be able to apply them. *e.g.*,:

(a) $\boldsymbol{F} = \left\langle e^{2xyz}, 0, 0\right\rangle$, S is the part of the cylinder $y^2+z^2 = 1$ with $0 \le x \le 2$.
answer: The vector field is only in the x direction, but this is the direction of the axis of the cylinder. Thus the surface normal of the cylinder is orthogonal to the field, *i.e.*, you are integrating 0 over the surface. Thus the answer is 0.

(b) $\boldsymbol{F} = \left\langle \frac{y}{b^2}, \frac{-x}{a^2}, 1\right\rangle$, S is the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$.
answer: I would go with the Divergence Theorem on this one; S is a closed surface, and $\nabla\cdot\boldsymbol{F} = 0$ everywhere, so the answer is 0.

(c) $\boldsymbol{F} = \langle x+z, x+y, \cos(x) - z\rangle$, S is the surface of some set D with volume V.

answer: Because D is given so ambiguously, we have to use the Divergence Theorem; S is a closed surface, and $\nabla \cdot \boldsymbol{F} = 1$ everywhere, so the answer is $1V$.

(d) $\boldsymbol{F} = \langle y - z, z - 2x, y \rangle$, S is the surface of the triangle with corners $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 3)$ with normal pointing away from the origin.
answer: You have no choice but to do this one directly. I think it is $-\frac{1}{3}$.

(e) $\boldsymbol{F} = \langle 2y, 2x, 2z \rangle$, S is the part of the cylinder $x^2 + y^2 = 1$ inside the sphere $x^2 + y^2 + (z - 4)^2 = 16$. Let S be oriented such that $\boldsymbol{n}$ points away from the z axis.
answer: I did this one directly, and got an answer of 0.

(f) $\boldsymbol{F} = \left\langle x^2, y^2, z^3 \right\rangle$, S is the surface of the cube bounded by $x = \pm 1, y = \pm 1, z = \pm 1$.
answer: We could use the Divergence Theorem, but $\nabla \cdot \boldsymbol{F}$ is not so nice; if you do use the DT, keep in mind the symmetry of the problem reduces to integrating $3z^2$ on the cube.
It may just be easier to evaluate this directly, because on each part of S (each of the 6 faces of the cube), the unit normal is one of $\pm\boldsymbol{i}, \pm\boldsymbol{j}, \pm\boldsymbol{k}$ So the answer should be $4 + -4 + 4 - 4 + 4 + 4 = 8$.

(g) $\boldsymbol{F} = \langle y, x + z, x \rangle$, S is the graph of $f(x, y) = x^2 + y$ over the region in the xy plane bounded by $0 \le x \le 1, 0 \le y \le 1 + x^2$, oriented such that $\boldsymbol{n} \cdot \boldsymbol{k} > 0$.
answer: You have no choice but to do this one directly. I think it is $-\frac{73}{30}$.

(h) $\boldsymbol{F} = \nabla \times \left\langle x^2 + y - 4, 3xy, 2xz + z^2 \right\rangle$, S is the surface of the hemisphere $x^2 + y^2 + z^2 = 16$ above the xy plane.
answer: I would use Stokes' Theorem *twice* to get this equivalent to integrating the flux of $\boldsymbol{F}$ over S', which is the circle $x^2 + y^2 = 16$ in the xy plane. Thus you only have to concern yourself with the $\boldsymbol{k}$ component of $\boldsymbol{F}$, which is $3y - 1$. Over the circle the $3y$ part disappears due to symmetry, and we are left with the integral of -1 over S', which is negative the area of S', *i.e.*, -16π.

(i) $\boldsymbol{F} = \langle y, -x, z \rangle$, S is the part of the parabola $z = x^2 + y^2$ below the plane $z = x$, oriented such that $\boldsymbol{n} \cdot \boldsymbol{k} < 0$. (May be difficult)
answer: I did this directly. There was some cancellation in the integral. I got $\frac{3}{32}\pi$. This one was not too bad–the hard parts were figuring out the domain to integrate over, and making a change to polar coordinates.

10. State the Divergence Theorem. State Stokes' Theorem.
answer: Look these up if you do not know them. You need to have these memorized, if not all the conditions (continuity, etc), but at least the equational "punchlines."

11. Compute a double or triple integral *i.e.*, compute

$$\iint_R f(x, y)\mathrm{d}A \quad \text{or} \quad \iiint_R f(x, y, z)\mathrm{d}V$$

by converting to (and evaluating) an iterated integral:

$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x, y)\mathrm{d}y\mathrm{d}x \quad \text{or} \quad \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

If necessary convert to polar, cylindrical, or spherical coordinates and correctly add the appropriate integrating factors.

(a) $f(x, y) = x + y$, R is bounded by $x = 0, y = x^2 - 1, y = 5 - x^2$.
answer: Flat out M20C, with no technique. I get $\frac{9}{2} + 8\sqrt{3}$, but I could be wrong.

(b) $f(x, y) = y \cos x$, R is bounded by $x = y^2, y = -1, y = 1, x = 3$.
answer: Mostly just M20C; by symmetry considerations the answer turns out to be 0.

(c) $f(x, y, z) = 1$, R is bounded by $x^2 + y^2 + z^2 = 4$, and $z \geq 1$.
answer: I think I set this one up in class as a cylindrical integral:

$$\int_0^{\sqrt{15}} \int_0^{2\pi} \int_1^{\sqrt{16-r^2}} 1 r \mathrm{d}z \mathrm{d}\theta \mathrm{d}r = 27\pi?$$

(d) $f(x, y, z) = z^2$, R is the region in the sphere $x^2 + y^2 + z^2 = 4$, and outside the cylinder $x^2 + y^2 = 1$.
answer: This one should be a cylindrical integral:

$$\int_1^2 \int_0^{2\pi} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} z^2 r \mathrm{d}z \mathrm{d}\theta \mathrm{d}r = \frac{12\pi\sqrt{3}}{5}?$$

(e) $f(x, y, z) = \sqrt{x^2 + y^2}$, R is the region $x^2 + y^2 \leq 4$, $4 - x^2 - y^2 \leq z \leq 4$.
answer: Another cylindrical integral:

$$\int_0^2 \int_0^{2\pi} \int_{4-r^2}^{4} \sqrt{r^2} r \mathrm{d}z \mathrm{d}\theta \mathrm{d}r = \frac{64\pi}{5}?$$

(f) $f(x, y, z) = 1$, R is the region $1 \leq x^2 + y^2 + z^2 \leq 4$, with $z > 0$.
answer: You could evaluate this as an integral, but it is easier to see it as half of the volume of the hollow sphere, which has volume $\frac{4\pi}{3}\left(2^3 - 1^3\right) = \frac{28\pi}{3}$, thus the answer is $\frac{14\pi}{3}$.

12. Given a function f and a region R, compute the integral

$$\iint_R f \mathrm{d}A \quad \text{or} \quad \iiint_R f \mathrm{d}V$$

by making a given change of variables and inserting the (determinant of the) Jacobian of the transformation. You should be able to compute the Jacobian for an "implicit" transformation (one in which u, v are given in terms of x, y and not the other way around). You should be able to figure out what the transformed region, $\tilde{R}$ looks like, and how to figure out its extents.

(a) $f(x, y, z) = x^2 - y^2$, where R is the region bounded by $y \leq x \leq y + 1$, and $-y \leq x \leq 1 - y$ by making the change $u = x + y, v = x - y$.
answer: Note there was a typo on the original handout, which had the bound $-y \leq x \leq 1 - x$. It has been corrected here. I think you should get

$$\int_0^1 \int_0^1 uv \frac{1}{2} \mathrm{d}u \mathrm{d}v = \frac{1}{8}$$

(b) $f(x,y,z) = \left[(x/a)^2 + (y/b)^2 + (z/c)^2\right]^k$, for some number k. Let R be the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1$. Make the transformation to u, v, w such that $\tilde{R}$ is a sphere.
answer: Use the transform $x = au, y = bv, z = cw$. Then you are trying to find

$$\iiint\limits_{\tilde{R}} \left[u^2 + v^2 + w^2\right]^k abc\mathrm{d}u\mathrm{d}v\mathrm{d}w.$$

Then transform to spherical coordinates to get

$$abc\int_0^1\int_0^{2\pi}\int_0^{\pi} \rho^{2k}\rho^2 \sin\phi \mathrm{d}\phi\mathrm{d}\theta\mathrm{d}\rho = abc\frac{4\pi}{2k+3}.$$

(c) $f(x,y,z) = \frac{3x}{2x-y+z}$, R is bounded by $0 \leq x - z \leq 1$, $0 \leq 2x + z \leq 1$, and $2 \leq 2x - y + z \leq 6$, by using an affine transform.
answer: By "affine transform," I mean let $u = x - z, v = 2x + z, w = 2x - y + z$. This simplifies the description of $\tilde{R}$: $0 \leq u \leq 1, 0 \leq v \leq 1, 2 \leq w \leq 6$. We find that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4.$$

Then we have to find

$$\int_0^1\int_0^1\int_2^6 \frac{u+v}{w}\frac{1}{4}\mathrm{d}w\mathrm{d}v\mathrm{d}u = \frac{\ln 3}{4}.$$

(d) $f(x,y,z) = x^2 + y^2$, R is a torus, $\tilde{R}$ is the rectangle $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq 2\pi$, and $0 \leq r \leq a$, under the transform:

$$\begin{aligned} x &= A\cos\phi + r\cos\phi\cos\theta \\ y &= A\sin\phi + r\sin\phi\cos\theta \\ z &= r\sin\theta, \end{aligned}$$

with $a \leq A$.
answer: The Jacobian calculation is a bit hairy; I think you get

$$\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = Ar + r^2\cos\theta.$$

Then our integral is

$$\int_0^a\int_0^{2\pi}\int_0^{2\pi} (A + r\cos\theta)^2 (A + r\cos\theta) r\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}r = 2\pi^2\left[A^3a^2 + \frac{3}{4}Aa^4\right].$$

(e) $f(x,y,z) = \frac{2y}{x+2y+4z}$, R is bounded by $1 \le x+2y+4z \le 4$, $0 \le y-2z \le 1$, $0 \le y+2z \le 2$, using an affine transform.
answer: By "affine transform," I mean let $u = x+2y+4z, v = y-2z, w = y+2z$. This simplifies the description of $\tilde{R}$: $1 \le u \le 4, 0 \le v \le 1, 0 \le w \le 2$. We find, again, that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4.$$

Then we have to find

$$\int_1^4 \int_0^1 \int_0^2 \frac{v+w}{u} \frac{1}{4} \mathrm{d}w \mathrm{d}v \mathrm{d}u = \frac{3\ln 4}{4}.$$

(f) $f(x,y) = e^{xy}$, R is bounded by $xy = 1, xy = 4, y = 1, y = 3$ using the change $x = u/v, y = v$. (A repeat from ex2 prep sheet, and stolen from Shenk's handout)
answer: This is Shenk's §15.6 worked problem #3. Answer is $\left(e^4 - e\right)\ln 3$.

(g) $f(x,y) = (x-y)^3$, R is bounded by $y = x, y = 3x, x = 3/2$, using the transform $u = x-y, v = 3x-y$. (A repeat from ex2 prep sheet)
answer: I still think the answer is $-\frac{243}{40}$.

13. Given some region in the plane, R, bounded by a curve C, use Green's Theorem in the plane to evaluate A, the area of R by computing a line integral of some field along C. Note that the field is likely not going to be conservative, unless R has zero area. *e.g.*,:
    (a) Let C be parametrized by $\boldsymbol{R}(t) = \langle \cos 3t \cos t, \cos 3t \sin t, 0 \rangle$, for $-\pi/6 \le t \le \pi/6$.
    answer: For each part of this question, it suffices to consider the function $\boldsymbol{F}(x,y,z) = \langle 0, x, 0 \rangle$, and to find the line integral $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{R}$. These integrals will make heavy use of the cosine and sine identities. For this part, I think the answer is $\frac{2\pi+3}{4}$.
    (b) Let C be parametrized by $\boldsymbol{R}(t) = \left\langle \cos^2 t, \cos t \sin t, 0 \right\rangle$, for $-\pi/2 \le t \le \pi/2$. (you should be able to solve this without any calculus, I believe)
    answer: I think the answer is $\frac{\pi}{4}$.
    (c) Let C be parametrized by $\boldsymbol{R}(t) = \left\langle \cos^3 t, \cos^2 t \sin t, 0 \right\rangle$, for $-\pi/2 \le t \le \pi/2$.
    answer: I think the answer is $\frac{3\pi}{16}$.
    (d) Let C be one loop of the lemniscate parametrized by $\boldsymbol{R}(t) = \left\langle \sqrt{\cos 2t} \cos t, \sqrt{\cos 2t} \sin t, 0 \right\rangle$ for $-\pi/4 \le t \le \pi/4$.
    answer: I think the answer is $\frac{1}{2}$.
14. Understand that the Laplacian is $\nabla^2 = \nabla \cdot \nabla$. Appreciate how Green's first formula is derived from the Divergence Theorem applied to the field $\phi \nabla \psi$.
answer: You need to know nothing more than the fact that $\nabla^2 = \nabla \cdot \nabla$, *i.e.*, that the Laplacian is the divergence of the gradient.