Solutions To Homework 1
Math 172 - Spring 2005

1. We classify the PDEs by looking at the coefficients of the second order terms. Written as $a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy}$, the classes are elliptic, parabolic, or hyperbolic if $b^2 - ac$ is less than, equal to, or greater than zero, respectively. We take the variables x and y to be real.
(1) $b^2 - ac = 0$; parabolic
(2) $b^2 - ac = -3/4$; elliptic
(3) $b^2 - ac = 5/4$; hyperbolic
(4) $b^2 - ac = 4x^4 + 1 > 0$ for all real x; hyperbolic
(5) $b^2 - ac = -x^2y^2$; when $x = 0$ or $y = 0$ it's parabolic, otherwise it's elliptic

2.(1) $\Delta u_3(x) = \Delta(u_1 - u_2) = \Delta u_1 - \Delta u_2 = f(x) - f(x) = 0$ for $x \in \Omega$, and
$u_3(x) = u_2(x) - u_1(x) = \phi_1(x) - \phi_2(x)$ for $x \in \partial\Omega$.
(2) It's given that $|\phi_3(x)| = |\phi_1(x) - \phi_2(x)| \le \varepsilon$ for $x \in \partial\Omega$, so by the properties of absolute value, $-\varepsilon \le \phi_3(x) \le \varepsilon$ for $x \in \partial\Omega$. Another way to say this is

$$\max_{x\in\partial\Omega} \phi_3(x) \le \varepsilon \qquad \text{and} \qquad \min_{x\in\partial\Omega} \phi_3(x) \ge \varepsilon$$

(3) Since $\Delta u_3(x) = 0$ for $x \in \Omega$, u_3 is harmonic on Ω. It's also continuous on $\overline{\Omega}$. These two properties allow us to apply the maximum/minimum principle: The maximum (minimum) value of $u_3(x)$ for $x \in \overline{\Omega}$ occurs for some $x \in \partial\Omega$. That is, the max/min occurs on the boundary. Thus,

$$\max_{x\in\overline{\Omega}} u_3(x) = \max_{x\in\partial\Omega} u_3(x) = \max_{x\in\partial\Omega} \phi_3(x) \le \varepsilon \qquad \text{and} \qquad \min_{x\in\overline{\Omega}} u_3(x) = \min_{x\in\partial\Omega} u_3(x) = \min_{x\in\partial\Omega} \phi_3(x) \ge \varepsilon$$

3. (1) Note that

$$x = \frac{\xi + \eta}{2} \qquad t = \frac{\xi - \eta}{2c}$$
$$\frac{\partial x}{\partial \xi} = \frac{1}{2} \qquad \frac{\partial t}{\partial \xi} = \frac{1}{2c}$$
$$\frac{\partial x}{\partial \eta} = \frac{1}{2} \qquad \frac{\partial t}{\partial \eta} = -\frac{1}{2c}$$

Now,

$$\frac{\partial \tilde{u}(\xi,\eta)}{\partial \xi} = \frac{\partial u(x,t)}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \xi} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{2c}\frac{\partial u}{\partial t}$$

and

$$\frac{\partial \tilde{u}(\xi,\eta)}{\partial \eta} = \frac{\partial u(x,t)}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \eta} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2c}\frac{\partial u}{\partial t} = 0$$

(2) The right hand side of the last equation is zero by the initial value data. Thus, $\tilde{u}(\xi,\eta)$ is a constant with respect to η, so we can write it as a function of just $\xi = x + ct$. That is, $\tilde{u}(\xi,\eta) = w(\xi) = w(x + ct)$.
(3) $\xi(x,t) = x + ct$ and $\eta(x,t) = x - ct$, so $\xi(x,0) = x$ and $\eta(x,0) = x$. Using the initial value data, we have

$$\phi(x) = u(x,0) = \tilde{u}\big(\xi(x,0), \eta(x,0)\big) = w\big(\xi(x,0)\big) = w(x)$$

so $u(x,t) = \tilde{u}(\xi,\eta) = w(\xi) = w(x+ct) = \phi(x+ct)$. This must be the unique solution since any other solution $u_2(x,t)$ must also equal $\phi(x + ct)$.
(4)

$$\max_{x\in\mathbb{R},t\ge0} |u_1(x,t) - u_2(x,t)| = \max_{x\in\mathbb{R},t\ge0} |\phi_1(x + ct) - \phi_2(x + ct)| = \max_{x\in\mathbb{R}} |\phi_1(x) - \phi_2(x)| \le \varepsilon$$

(5) $u(x,t)$ remains constant on lines of the form $x_0 = x + ct$. So if $u(x_0, 0) = \phi(x_0)$, then $u(x,t) = \phi(x + ct) = \phi(x_0)$ also. The initial values of u propagate to the left at a rate of c.
4.

Note: The IVP (1.5.6) used in exercise 1.4 represents a 1-dimensional wave that can travel infinitely in either direction. The IVP (1.1.4) used in exercise 1.5 and 1.6 represents a 1-dimensional wave that is contained between two endpoints ($x = 0$ and $x = l$).

(1.4) By exercise 1.3, a solution must be of the form $u(x,t) = w_1(x+ct)+w_2(x-ct)$. Note that $\frac{\partial w_1(x+ct)}{\partial t} = cw_1'(x+ct)$ and $\frac{\partial w_2(x-ct)}{\partial t} = -cw_2'(x - ct)$ The initial conditions give

$$u(x,0) = w_1(x) + w_2(x) = u_0(x) \tag{1}$$

$$u_t(x,0) = c\big(w_1'(x) - w_2'(x)\big) = u_1(x) \tag{2}$$

Applying the fundamental theorem of calculus to the last equation gives

$$w_1(x) - w_2(x) = \frac{1}{c}\int_0^x u_1(s)\,ds \tag{3}$$

Now, adding (1) to (3) gives

$$2w_1(x) = u_0(x) + \frac{1}{c}\int_0^x u_1(s)\,ds \tag{4}$$

and subtracting (3) from (1) gives

$$2w_2(x) = u_0(x) - \frac{1}{c}\int_0^x u_1(s)\,ds \tag{5}$$

Substituting $x+ct$ in (4) and $x-ct$ in (5) for x, adding the two equations and dividing by two gives

$$w_1(x+ct) + w_2(x-ct) = \frac{1}{2}\left[u_0(x+ct) + u_0(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} u_1(s)\,ds\right]$$

We started with the initial conditions and ended up here, so this must be *the* solution to the IVP.

(1.5)

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{d}{dt}\left\{\frac{1}{2}\int_0^l \left[\frac{1}{c^2}u_t(x,t)^2 + u_x(x,t)^2\right]dx\right\}\\
&= \frac{1}{2}\int_0^l \left[\frac{1}{c^2}\frac{d}{dt}\left(u_t(x,t)^2\right) + \frac{d}{dt}\left(u_x(x,t)^2\right)\right]dx\\
&= \frac{1}{2}\int_0^l \left[\frac{1}{c^2}2u_t(x,t)u_{tt}(x,t) + 2u_x(x,t)u_{xt}(x,t)\right]dx \qquad \text{(by the chain rule)}\\
&= \int_0^l \left[\frac{1}{c^2}u_t(x,t)u_{tt}(x,t) + u_x(x,t)u_{xt}(x,t)\right]dx
\end{aligned}$$

Now do integration by parts on the second term of the integrand using $dw = u_{xt}\,dx$, $w = u_t$, $v = u_x$, $dv = u_{xx}\,dx$ to get

$$\begin{aligned}
\frac{d}{dt}E(t) &= u_t(x,t)u_x(x,t)\Big|_{x=0}^{l} + \int_0^l \left[\frac{1}{c^2}u_t(x,t)u_{tt}(x,t) - u_t(x,t)u_{xx}(x,t)\right]dx\\
&= u_t(x,t)u_x(x,t)\Big|_{x=0}^{l} + \int_0^l \left[u_t(x,t)\Big(\underbrace{\frac{1}{c^2}u_{tt}(x,t) - u_{xx}(x,t)}_{=0}\Big)\right]dx\\
&= u_t(x,t)u_x(x,t)\Big|_{x=0}^{l}\\
&= u_t(l,t)u_x(l,t) - u_t(0,t)u_x(0,t)
\end{aligned}$$

(1.6) The condition $u(0,t) = u(l,t) = 0$ for all $t \geq 0$ can be thought of as the fixed endpoints of a string. Obviously, if they're fixed at the value zero, then they don't change with time, or in terms of calculus $u_t(0,t) = u_t(l,t) = 0$ for all $t \geq 0$. Putting this in the result of exercise 1.5 gives $\frac{d}{dt}E(t) = 0$ for all $t \geq 0$.

Now we have one condition that says $u_t(x,0) = 0$ for $0 \leq x \leq l$. The other condition, $u(x,0) = 0$ for $0 \leq x \leq l$ implies $u_x(x,0) = 0$ for $0 \leq x \leq l$ also. Putting these in the integral for the energy at time $t = 0$ gives $E(0) = 0$. And since $\frac{d}{dt}E(t) = 0$ we must have $E(t) = E(0) = 0$ for all $t \geq 0$.

(1.10) By exercise 1.9, if $u = v(r)$ depends only on the distance r between x and a fixed point, then we just need to solve the (ordinary) differential equation

$$\Delta u = \Delta v(r) = v''(r) + \frac{n-1}{r} v'(r) = 0 \tag{6}$$

Multiplying by the integrating factor $\exp\left(\int \frac{n-1}{r}\, dr\right) = \exp\left((n-1)\log r\right) = \exp\left(\log r^{n-1}\right) = r^{n-1}$ gives

$$r^{n-1}v''(r) + (n-1)r^{n-2}v'(r) = \left[r^{n-1}v'(r)\right]' = 0$$

Integrating this gives

$$\begin{aligned} r^{n-1}v'(r) &= C \\ v'(r) &= Cr^{1-n} \end{aligned}$$

Integrating this gives

$$v(r) = \begin{cases} Cr + B, & n = 1 \\ C\log r + B, & n = 2 \\ \hat{C}r^{2-n} + B, & n \geq 3 \end{cases}$$

as the general solution. ($\hat{C} = C/(2-n)$ is just another constant.)