Solutions To Homework 2
Math 172 - Spring 2005

2. Using Taylor's theorem we can write

$$u(x+h_1,y) = u(x,y) + h_1 u_x(x,y) + \frac{h_1^2}{2} u_{xx}(x,y) + \mathcal{O}(h_1^3) \tag{1}$$

$$u_1 = u_0 + h_1 u_x(x,y) + \frac{h_1^2}{2} u_{xx}(x,y) + \mathcal{O}(h_1^3) \tag{2}$$

$$\frac{u_1}{h_1} = \frac{u_0}{h_1} + u_x(x,y) + \frac{h_1}{2} u_{xx}(x,y) + \mathcal{O}(h_1^2) \tag{3}$$

Similarly, we can expand $u_3 = u(x-h_3,y)$ in terms of $u_0 = u(x,y)$:

$$u(x-h_3,y) = u(x,y) - h_3 u_x(x,y) + \frac{(-h_3)^2}{2} u_{xx}(x,y) + \mathcal{O}(h_3^3) \tag{4}$$

$$u_3 = u_0 - h_3 u_x(x,y) + \frac{h_3^2}{2} u_{xx}(x,y) + \mathcal{O}(h_3^3) \tag{5}$$

$$\frac{u_3}{h_3} = \frac{u_0}{h_3} - u_x(x,y) + \frac{h_3}{2} u_{xx}(x,y) + \mathcal{O}(h_3^2) \tag{6}$$

Now if we add (3) and (6) we'll eliminate the $u_x(x,y)$ term to get an approximation for $u_{xx}(x,y)$ in term of $u_0 = u(x,y)$, u_1 and u_3.

$$\frac{u_1}{h_1} + \frac{u_3}{h_3} = \frac{u_0}{h_1} + \frac{u_0}{h_3} + \frac{h_1+h_3}{2} u_{xx}(x,y) + \mathcal{O}(h_1^2) + \mathcal{O}(h_3^2) \tag{7}$$

$$-\frac{h_1+h_3}{2} u_{xx}(x,y) = \frac{u_0-u_1}{h_1} + \frac{u_0-u_3}{h_3} + \mathcal{O}(h_1^2) + \mathcal{O}(h_3^2) \tag{8}$$

$$u_{xx}(x,y) = -\frac{2}{h_1+h_3}\left(\frac{u_0-u_1}{h_1} + \frac{u_0-u_3}{h_3}\right) + \mathcal{O}(h_1) + \mathcal{O}(h_3) \tag{9}$$

$$u_{xx}(x,y) = -\frac{2}{h_1 h_3} u_0 + \frac{2}{h_1(h_1+h_3)} u_1 + \frac{2}{h_3(h_1+h_3)} u_3 + \mathcal{O}(h_1) + \mathcal{O}(h_3) \tag{10}$$

Similarly,

$$u_{yy}(x,y) = -\frac{2}{h_2 h_4} u_0 + \frac{2}{h_2(h_2+h_4)} u_2 + \frac{2}{h_2(h_2+h_4)} u_4 + \mathcal{O}(h_2) + \mathcal{O}(h_4) \tag{11}$$

and we finally arrive at the approximation

$$u_{xx}(x,y) + u_{yy}(x,y) \approx -2\left(\frac{1}{h_1 h_3} + \frac{1}{h_2 h_4}\right) u_0 + \frac{2}{h_1(h_1+h_3)} u_1 + \frac{2}{h_2(h_2+h_4)} u_2 + \frac{2}{h_3(h_1+h_3)} u_3 + \frac{2}{h_2(h_2+h_4)} u_4$$

(b) If $h_1 = h_3 = h$ and $h_2 = h_4 = k$, the approximation reduces to

$$u_{xx}(x,y) + u_{yy}(x,y) \approx -2\left(\frac{1}{h^2} + \frac{1}{k^2}\right) u_0 + \frac{1}{h^2} u_1 + \frac{1}{k^2} u_2 + \frac{1}{h^2} u_3 + \frac{1}{k^2} u_4 \tag{12}$$

$$= \frac{u_1 - 2u_0 + u_3}{h^2} + \frac{u_2 - 2u_0 + u_4}{k^2} \tag{13}$$

Compare this to equation (2.1.13) in the notes.

(c) If $h_1 = h_2 = h_3 = h_4 = h$, we get the usual five point star difference formula for the Laplacian (2.1.12 in the notes):

$$u_{xx}(x,y) + u_{yy}(x,y) \approx \frac{-4u_0 + u_1 + u_2 + u_3 + u_4}{h^2}$$

3.(5) Let $z_- = e^{i(\theta_k - \theta_l)} = e^{i(k-l)\pi/(n+1)}$ and $z_+ = e^{i(\theta_k+\theta_l)} = e^{i(k+l)\pi/(n+1)}$. Then

$$z_-^{n+1} = e^{i(\theta_k - \theta_l)(n+1)} = e^{i(k-l)\pi} = \cos(k-l)\pi + i\sin(k-l)\pi = \begin{cases} -1, & \text{if } k-l \text{ is odd} \\ +1, & \text{if } k-l \text{ is even} \end{cases} \tag{14}$$

and

$$z_+^{n+1} = e^{i(\theta_k+\theta_l)(n+1)} = e^{i(k+l)\pi} = z_-^{n+1} e^{i2l\pi} = z_-^{n+1} \tag{15}$$

because $e^{i2l\pi} = \cos(2l\pi) + i\sin(2l\pi) = 1$ for any integer l. If $k-l$ is even and nonzero (so that $z_- \neq 1$), then apply part (4) and equation (14) to get

$$\sum_{j=1}^{n} z_-^j = \sum_{j=1}^{n} z_+^j = \frac{z_\pm^{n+1} - z_\pm}{z_\pm - 1} = \frac{1 - z_\pm}{z_\pm - 1} = -1$$

(6) Rationalize the denominator:

$$\frac{z_+ + 1}{z_+ - 1} = \frac{z_+ + 1}{z_+ - 1} \cdot \frac{\overline{z_+} - 1}{\overline{z_+} - 1} = \frac{z_+\overline{z_+} + z_+ - \overline{z_+} - 1}{z_+\overline{z_+} - z_+ - \overline{z_+} + 1} = \frac{|z_+|^2 - 1 + 2i\Im(z_+)}{|z_+|^2 - 2\Re(z_+) + 1} \tag{16}$$

where $\Im$ denotes the imaginary part of a complex number: $\Im(a+ib) = b$. Since $|z_+|^2 = z_+\overline{z_+} = e^{i(\theta_k-\theta_l)}e^{-i(\theta_k-\theta_l)} = 1$ equation (16) becomes

$$\frac{z_+ + 1}{z_+ - 1} = \frac{2i\Im(z_+)}{2 - 2\Re(z_+)} = i\frac{\Im(z_+)}{1 - \Re(z_+)}$$

a strictly imaginary number. Therefore, $\Re\left[\frac{z_+ + 1}{z_+ - 1}\right] = 0$, and similarly for z_-.

Now, if $k-l$ is odd, then $z_-^{n+1} = z_+^{n+1} = -1$ by (14). Apply parts (3) and (4) to get

$$\mathbf{v_k} \cdot \mathbf{v_l} = \Re\left[\frac{z_+^{n+1} - z_+}{z_+ - 1} - \frac{z_-^{n+1} - z_-}{z_- - 1}\right] = \Re\left[\frac{-1 - z_+}{z_+ - 1} - \frac{-1 - z_-}{z_- - 1}\right] = \Re\left[\frac{z_- + 1}{z_- - 1} - \frac{z_+ + 1}{z_+ - 1}\right] = 0 \tag{17}$$

(7) (Bonus)

$$\mathbf{v_k} \cdot \mathbf{v_k} = \frac{1}{2}\Re\left[\sum_{j=1}^{n} e^{ij(\theta_k-\theta_k)} - \sum_{j=1}^{n} e^{ij(\theta_k+\theta_k)}\right] = \frac{1}{2}\Re\left[\sum_{j=1}^{n} e^0 - \sum_{j=1}^{n} e^{ij2\theta_k}\right] = \frac{1}{2}\Re\left[\sum_{j=1}^{n} 1 - (-1)\right] = \frac{n+1}{2} \tag{18}$$

where $\sum_{j=1}^{n} e^{ij2\theta_k} = -1$ by using the result of part (5).