Solutions To Homework 3
Math 172 - Spring 2005

1. (3.3 from notes) See Theorem 3.1.1 and the discussion afterward leading to Theorem 3.1.2. By Theorem 3.1.2, for $r = \Delta t/(\Delta x)^2 = 1/6$ and forcing function satisfying $f_t + f_{xx} = 0$, the maximum error in the computed solution is bounded:

$$\max_{i,j} |u(x_i, t_j)| \leq B \cdot T \cdot \big((\Delta t)^2 + (\Delta x)^4\big) \leq \frac{37}{36} B \cdot T \cdot (\Delta x)^4 \tag{1}$$

where $0 < t_j < T$ and B depends only on u_{ttt} and u_{xxxxxx}. The last inequality comes by substituting $\Delta t = (\Delta x)^2/6$.

In our case, the forcing function $f(x,t) = 0$, so for $r = 1/6$ we expect the computed solution to be accurate to order $\mathcal{O}\big((\Delta x)^4\big)$.

2.

$$\begin{aligned}\delta_x^- \mathcal{F}_n^j &= \frac{1}{h}\left[\mathcal{F}_n^j - \mathcal{F}_{n-1}^j\right] = \frac{1}{h}\left[\lambda^j e^{i\beta nh} - \lambda^j e^{i\beta(n-1)h}\right] = \frac{\lambda^j e^{i\beta nh}}{h}\left[1 - e^{-i\beta h}\right] \\ &= \frac{\mathcal{F}_n^j}{h}\left[1 - \cos(-\beta h) - i\sin(-\beta h)\right] = \frac{\mathcal{F}_n^j}{h}\left[1 - \cos(\beta h) + i\sin(\beta h)\right]\end{aligned}$$

$$\begin{aligned}\delta_x \mathcal{F}_n^j &= \frac{1}{2h}\left[\mathcal{F}_{n+1}^j - \mathcal{F}_{n-1}^j\right] = \frac{1}{2h}\left[\lambda^j e^{i\beta(n+1)h} - \lambda^j e^{i\beta(n-1)h}\right] = \frac{\lambda^j e^{i\beta nh}}{2h}\left[e^{i\beta h} - e^{-i\beta h}\right] \\ &= \frac{\mathcal{F}_n^j}{2h}\left[\cos(\beta h) + i\sin(\beta h) - \cos(-\beta h) - i\sin(-\beta h)\right] \\ &= \frac{\mathcal{F}_n^j}{2h}\left[\cos(\beta h) + i\sin(\beta h) - \cos(\beta h) + i\sin(\beta h)\right] = \frac{\mathcal{F}_n^j}{2h}\left[2i\sin(\beta h)\right] = \frac{\mathcal{F}_n^j}{h} i\sin(\beta h)\end{aligned}$$

$$\begin{aligned}\delta_{xx} \mathcal{F}_n^j &= \frac{1}{h^2}\left[\mathcal{F}_{n+1}^j - 2\mathcal{F}_n^j + \mathcal{F}_{n-1}^j\right] = \frac{1}{h^2}\left[\lambda^j e^{i\beta(n+1)h} - 2\lambda^j e^{i\beta nh} + \lambda^j e^{i\beta(n-1)h}\right] \\ &= \frac{\lambda^j e^{i\beta nh}}{h^2}\left[e^{i\beta h} - 2 + e^{-i\beta h}\right] = \frac{\mathcal{F}_n^j}{h^2}\left[e^{i\beta h/2} - e^{-i\beta h/2}\right]^2 \\ &= \frac{\mathcal{F}_n^j}{h^2}\left[2i\sin(\beta h/2)\right]^2 = \frac{\mathcal{F}_n^j}{h^2}\left[-4\sin^2(\beta h/2)\right]\end{aligned}$$

$$\delta_t^+ \mathcal{F}_n^j = \frac{1}{k}\left[\mathcal{F}_n^{j+1} - \mathcal{F}_n^j\right] = \frac{1}{k}\left[\lambda^{j+1} e^{i\beta nh} - \lambda^j e^{i\beta nh}\right] = \frac{\lambda^j e^{i\beta nh}}{k}\left[\lambda - 1\right] = \frac{\mathcal{F}_n^j}{k}\left[\lambda - 1\right]$$

$$\delta_t^- \mathcal{F}_n^j = \frac{1}{k}\left[\mathcal{F}_n^j - \mathcal{F}_n^{j-1}\right] = \frac{1}{k}\left[\lambda^j e^{i\beta nh} - \lambda^{j-1} e^{i\beta nh}\right] = \frac{\lambda^j e^{i\beta nh}}{k}\left[1 - \lambda^{-1}\right] = \frac{\mathcal{F}_n^j}{k}\left[1 - 1/\lambda\right]$$

$$\delta_t \mathcal{F}_n^j = \frac{1}{k}\left[\mathcal{F}_n^{j+1} - \mathcal{F}_n^{j-1}\right] = \frac{1}{k}\left[\lambda^{j+1} e^{i\beta nh} - \lambda^{j-1} e^{i\beta nh}\right] = \frac{\lambda^j e^{i\beta nh}}{k}\left[\lambda - \lambda^{-1}\right] = \frac{\mathcal{F}_n^j}{k}\left[\lambda - 1/\lambda\right]$$

$$\delta_{tt} \mathcal{F}_n^j = \frac{1}{k}\left[\mathcal{F}_n^{j+1} - 2\mathcal{F}_n^j + \mathcal{F}_n^{j-1}\right] = \frac{1}{k}\left[\lambda^{j+1} e^{i\beta nh} - 2\lambda^j e^{i\beta nh} + \lambda^{j-1} e^{i\beta nh}\right] = \frac{\lambda^j e^{i\beta nh}}{k}\left[\lambda - 2 + \lambda^{-1}\right] = \frac{\mathcal{F}_n^j}{k}\left[\lambda - 2 + 1/\lambda\right]$$