Solutions To Homework 4
Math 172 - Spring 2005

**2.** For $U_n^j = \lambda^j e^{i\beta nh}$ we have

$$\begin{aligned}\delta_x^2 U_n^j &= U_{n+1}^j - 2U_n^j + U_{n-1}^j = \lambda^j e^{i\beta(n+1)h} - 2\lambda^j e^{i\beta nh} + \lambda^j e^{i\beta(n-1)h} = \lambda^j e^{i\beta nh}(e^{i\beta h} - 2 + e^{-i\beta h}) \\ &= U_n^j\big(-4\sin^2(\beta h/2)\big)\end{aligned}$$

Similarly,

$$\begin{aligned}\delta_x^2 U_n^{j+1} &= U_n^j \lambda\big(-4\sin^2(\beta h/2)\big) \\ \delta_x^2 U_n^{j-1} &= U_n^j \frac{1}{\lambda}\big(-4\sin^2(\beta h/2)\big)\end{aligned}$$

**(1)** Inserting $U_n^j = \lambda^j e^{i\beta nh}$ into the scheme we get

$$\begin{aligned}\lambda U_n^j - \frac{1}{\lambda}U_n^j &= \frac{s}{3}U_n^j\big(-4\sin^2(\beta h/2)\big)\left[\lambda + 1 + \frac{1}{\lambda}\right] \\ \lambda^2 U_n^j - U_n^j &= \frac{s}{3}U_n^j\big(-4\sin^2(\beta h/2)\big)\left[\lambda^2 + \lambda + 1\right] \\ \lambda^2 - 1 &= \frac{s}{3}\big(-4\sin^2(\beta h/2)\big)\left[\lambda^2 + \lambda + 1\right] \\ \lambda^2 - 1 &= \alpha\left[\lambda^2 + \lambda + 1\right] \\ 0 &= (\alpha - 1)\lambda^2 + \alpha\lambda + (\alpha + 1) \\ 0 &= \lambda^2 + \frac{\alpha}{\alpha - 1}\lambda + \frac{\alpha + 1}{\alpha - 1}\end{aligned}$$

where $\alpha = \frac{s}{3}\big(-4\sin^2(\beta h/2)\big) \le 0$. We need to show that $|\lambda| \le 1$. We do this using the "amazing fact" involving roots of the equation $z^2 + bz + c = 0$. Since $\alpha \le 0$ we have

$$\begin{array}{rcccl} 1 - 2\alpha & \ge & 1 & \ge & -1 \\ 1 - \alpha & \ge & \alpha + 1 & \ge & \alpha - 1 \\ -1 & \le & \dfrac{\alpha + 1}{\alpha - 1} & \le & 1 \end{array}$$

(The inequality flips because $\alpha - 1 \le 0$.) Thus, $|c| = \left|\frac{\alpha+1}{\alpha-1}\right| \le 1$. We also have

$$\begin{array}{rcccl} -3\alpha & \ge & 0 & \ge & \alpha \\ -2\alpha & \ge & \alpha & \ge & 2\alpha \\ -(\alpha - 1) - (\alpha + 1) & \ge & \alpha & \le & (\alpha - 1) + (\alpha + 1) \\ -1 - \dfrac{\alpha + 1}{\alpha - 1} & \le & \dfrac{\alpha}{\alpha - 1} & \le & 1 + \dfrac{\alpha + 1}{\alpha - 1} \\ -c & \le & b & \le & c \end{array}$$

Therefore, the two roots are on or within the unit circle in the complex plane (i.e. $|\lambda| \le 1$).

**(2)** For this scheme we have

$$\begin{aligned}\left(\lambda - \frac{1}{\lambda}\right)U_n^j &= U_n^j \frac{s}{6}\big(-4\sin^2(\beta h/2)\big)\left[\lambda + 4 + \frac{1}{\lambda}\right] \\ \lambda - \frac{1}{\lambda} &= \alpha\left[\lambda + 4 + \frac{1}{\lambda}\right] \\ \lambda^2 - 1 &= \alpha\left[\lambda^2 + 4\lambda + 1\right] \\ 0 &= (\alpha - 1)\lambda^2 + 4\alpha\lambda + (\alpha + 1) \\ 0 &= \lambda^2 + \frac{4\alpha}{\alpha - 1}\lambda + \frac{\alpha + 1}{\alpha - 1}\end{aligned}$$

where $\alpha = \dfrac{s}{6}\big(-4\sin^2(\beta h/2)\big)$. In this case, $|c| = \left|\frac{\alpha+1}{\alpha-1}\right| \le 1$ again, but

$$\begin{aligned}2\alpha &\le 0 \\ 4\alpha &\le 2\alpha = (\alpha - 1) + (\alpha + 1) \\ b = \frac{4\alpha}{\alpha - 1} &\ge 1 + \frac{\alpha + 1}{\alpha - 1} = 1 + c\end{aligned}$$

Thus at least one root $|\lambda| > 1$ and the scheme is unstable for any $s$.