Solutions To Homework 5
Math 172 - Spring 2005

1. By Taylor's Theorem,

$$g(x+(k+1)\Delta x) = g(x) + (k+1)(\Delta x)g'(x) + \frac{(k+1)^2(\Delta x)^2}{2}g''(x) + \mathcal{O}((k+1)^3(\Delta x)^3)$$
$$g(x+k\Delta x) = g(x) + k(\Delta x)g'(x) + \frac{k^2(\Delta x)^2}{2}g''(x) + \mathcal{O}(k^3(\Delta x)^3)$$

Subtracting the second equation from the first we get

$$g(x+(k+1)\Delta x) - g(x+k\Delta x) = (\Delta x)g'(x) + \frac{[(k+1)^2-k^2](\Delta x)^2}{2}g''(x) + \mathcal{O}((k+1)^3(\Delta x)^3) + \mathcal{O}(k^3(\Delta x)^3)$$
$$\frac{g(x+(k+1)\Delta x) - g(x+k\Delta x)}{\Delta x} = g'(x) + \frac{[2k+1]\Delta x}{2}g''(x) + \mathcal{O}((k+1)^3(\Delta x)^2) + \mathcal{O}(k^3(\Delta x)^2)$$
$$\frac{g(x+(k+1)\Delta x) - g(x+k\Delta x)}{\Delta x} = g'(x) + \mathcal{O}\big((k+1/2)\Delta x\big)$$
$$\frac{g(x+(k+1)\Delta x) - g(x+k\Delta x)}{\Delta x} = g'(x) + \mathcal{O}(k\Delta x)$$

2.(1) $u(x,0) = u_0(x - x\cdot 0) = u_0(x)$ so it satisfies the initial condition. Using the chain rule, $u_t = u_0'(x-xt)(-x)$ and $u_x = u_0'(x-xt)(1-t)$. Plug these into the PDE and equality is satisfied.
(2) Since $u(x,1) = u_0(x - x\cdot 1) = u_0(0)$ for all $x \in (-\infty,\infty)$, we see that at time $t=1$ the solution is a constant in x.
(3) As $t\to\infty$, the speed $a(x,t)\to\pm\infty$ (depending on the sign of x). If we write the PDE as $(1-t)u_t + xu_x = 0$, we see that as $t\to 1$ we'll have $u_x \approx 0$.
(4) For $x\in[-1,1]$ and $t\in[0,1/2]$,

$$|a| = \left|\frac{x}{1-t}\right| \le \frac{1}{|1-t|} \le \frac{1}{1-1/2} = 2$$

so we need $\Delta t/\Delta x \le 1/2$ to have $|\nu| = |a\Delta t/\Delta x| \le 1$.
(5) (a) $a(\xi(t),t) = \dfrac{k/(1-t)}{1-t} = \dfrac{k}{(1-t)^2}$ and $\xi'(t) = \dfrac{k}{(1-t)^2}$.

(b) Using part (1) we have $u(\xi(t),t) = u_0(\xi(t)(1-t)) = u_0\left(\dfrac{k}{1-t}(1-t)\right) = u_0(k)$, so u is a constant along the characteristic curve.
(6)