2005S M172 Exam 1 Preparation

The first midterm exam is Friday, April 29, during the class period.

The following formula will be provided on your exam:

$$1 - \cos\theta = 2\sin^2(\theta/2)$$

Everything else must be committed to memory.

The following look like good exam type questions and material.
1. What does it mean for a problem to be "well posed" for a given PDE? State a well-posed problem, and one which is not well posed.
2. Give the closed form solution to the transport equation:

$$\begin{cases} u_t + a u_x = 0 & x \in \mathbb{R},\, t > 0 \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases}$$

What are the characteristics of this equation?
3. Give the closed form solution to the wave equation:

$$\begin{cases} u_{xx} - \frac{1}{c^2} u_{tt} = 0 & x \in \mathbb{R},\, t > 0 \\ u(x,0) = \phi(x) & x \in \mathbb{R}, \\ u_t(x,0) = \psi(x) & x \in \mathbb{R}, \end{cases}$$

What are the characteristics of this equation?
4. Give the classification scheme for second order quasilinear PDEs of the type:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + DU_x + EU_y + FU + G = 0$$

Classify the following PDE's as hyperbolic, parabolic, or elliptic, renaming t to be y if necessary:

1. $u_{xx} - u_{yy} = 0$.
2. $u_{xx} + u_{yy} = 0$.
3. $u_t - u_{xx} = f$.
4. $u_t - u_x = f$.
5. $u_{xx} - \frac{1}{c^2} u_{tt} = 0$.
6. $u_{xx} + u u_{xy} + u_y y = f$.

5. State Taylor's Theorem for $f(x+h)$. Use this theorem to get a $\mathcal{O}(h)$ approximation of $f'(x)$. Use it to get a $\mathcal{O}(h^2)$ approximation to $f''(x)$.
6. Use Taylor's Theorem to find the form of the following differences:

$$\begin{aligned} \frac{f(x+h) - f(x-h)}{2h} &= f'(x) + a_1 h^2 f'''(x) + \mathcal{O}(h^4) \\ \frac{f(x+2h) - f(x-2h)}{4h} &= f'(x) + a_2 h^2 f'''(x) + \mathcal{O}(h^4) \end{aligned}$$

Combine these differences to get a $\mathcal{O}(h^4)$ difference approximation, *i.e.*, find β_1, β_2 such that

$$\beta_1 \frac{f(x+h) - f(x-h)}{2h} + \beta_2 \frac{f(x+2h) - f(x-2h)}{4h} = f'(x) + \mathcal{O}(h^4)$$

7. What does it mean for $\boldsymbol{v}, \lambda$ to be eigenvector and eigenvalue of matrix A? What are the eigenvalues and -vectors of the identity matrix I? Suppose $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of matrix A. What are the eigenvalues of $\mathsf{B} = \alpha\mathsf{I} + \beta\mathsf{A}$?

8. Let $\{\boldsymbol{v_i}\}_{i=1}^n$ be the eigenvectors of matrix A. Suppose these vectors span $\mathbb{R}^n$. Suppose the eigenvalues of A are all nonnegative and real. What can you say about

$$\boldsymbol{x}^\top \mathsf{A}\boldsymbol{x}$$

for arbitrary vector $\boldsymbol{x}$?

9. Suppose the eigenvalues of a matrix are all positive and real. Can the matrix be singular? Suppose A is a singular matrix; can you name one of its eigenvalues?

10. What are the eigenvalues and corresponding eigenvectors of the matrix

$$\mathsf{B} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix}$$

11. What are the different errors that might occur in a finite difference scheme? Do you think they can be minimized independently? How do you define consistency? Stability?

12. Give the explicit scheme for the solution of the Parabolic nonhomogeneous heat equation with boundary values:

$$\begin{cases} u_t - u_{xx} = f(x,t) & x \in [0,1]\,,\ t \in [0,T]\,, \\ u(x,0) = u_0(x) & x \in [0,1]\,, \\ u(0,t) = a(t) & t \in (0,T]\,, \\ u(1,t) = b(t) & t \in (0,T] \end{cases}$$

Write the scheme as

$$\boldsymbol{U^{j+1}} = \mathsf{A}\boldsymbol{U^j} + \boldsymbol{F}^j$$

for some matrix A, and some starting vector $\boldsymbol{U^0}$. What are the eigenvectors,-values of this matrix?

13. Perform a stability analysis for this explicit scheme. Suppose that some error occurs at the first step, and your program starts with $\tilde{\boldsymbol{U}}^{\boldsymbol{0}}$, then computes $\tilde{\boldsymbol{U}}^{\boldsymbol{j}}$ flawlessly according to this scheme. Let $\boldsymbol{Z^j} = \boldsymbol{U^j} - \tilde{\boldsymbol{U}}^{\boldsymbol{j}}$. Give a finite difference scheme for $\boldsymbol{Z^j}$. Find the necessary and sufficient condition for stability, *i.e.*, find conditions on $\Delta t, \Delta x$ such that $\left\|\boldsymbol{Z^j}\right\| \to 0$, as $j \to \infty$.

You should be prepared to do this three ways:

1. Directly using a "sup-norm" argument: let $z^j = \max_i \left|Z_i^j\right|$. Find conditions such that $z^{j+1} \leq z^j$.
2. Using an eigenvalue argument: find conditions such that the eigenvalues of A are sufficiently small.
3. By Fourier Analysis: pretend that U_j^n takes the form $\lambda^n e^{i\beta j \Delta x}$, and find conditions such that $|\lambda|$ is sufficiently small.

14. Give the implicit scheme, the θ-scheme, and the Crank-Nicolson scheme for the heat equation. Perform stability analyses for these by the Fourier method.

15. (textbook 3.9) Consider the homogeneous heat equation with homogeneous boundary conditions:

$$\begin{cases} u_t = u_{xx} & x \in [0,1]\,, t \in [0,T]\,, \\ u(x,0) = u_0(x) & x \in [0,1]\,, \\ u(0,t) = 0 & t \in (0,T]\,, \\ u(1,t) = 0 & t \in (0,T] \end{cases}$$

Suppose U_l^j is the solution found by the Crank-Nicolson scheme. That is

$$\begin{cases} U_l^0 = u_0(x_l) & 0 < l < m \\ U_0^j = 0 & 0 \le j \le n \\ U_m^j = 0 & 0 \le j \le n \\ 2(1+s)U_l^{j+1} - s\left[U_{l+1}^{j+1} + U_{l-1}^{j+1}\right] = 2(1-s)U_l^j + s\left[U_{l+1}^j + U_{l-1}^j\right] & 0 < l < m,\, 0 < j \le n \end{cases}$$

Furthermore, suppose that U_l^j is *separable*, *i.e.*, there is some $\{X_l\}_{l=0}^m$ and $\{T_j\}_{j=0}^n$ such that

$$U_l^j = X_l T_j$$

Show that $T_{j+1} = \lambda T_j$ for some λ, and thus $T_j = \lambda^j T_0$.

16. Set up an explicit scheme to solve *Burgers' Equation*:

$$\begin{cases} -uu_x + \nu u_{xx} = u_t & x \in [0,1]\,, t \in [0,T]\,, \\ u(x,0) = u_0(x) & x \in [0,1]\,, \\ u(0,t) = a(t) & t \in (0,T]\,, \\ u(1,t) = b(t) & t \in (0,T] \end{cases}$$

Use the centered difference approximation to u_x, and the forward difference for u_t. Do you get a linear system?

17. Set up the method of lines solution to the *Burgers' Equation*:

$$\begin{cases} -uu_x + \nu u_{xx} = u_t & x \in [0,1]\,, t \in [0,T]\,, \\ u(x,0) = u_0(x) & x \in [0,1]\,, \\ u(0,t) = a(t) & t \in (0,T]\,, \\ u(1,t) = b(t) & t \in (0,T] \end{cases}$$

Divide the unit interval into M pieces, and set up a system of ODEs for the functions $\{U_l(t)\}_{l=0}^M$, which are supposed to approximate $u(x_l,t) = u(l/M,t)$, with u the solution to the PDE. This is a difficult system, do not attempt to solve it.

18. (Morton & Mayers, Exercise 2.7) Consider the PDE

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) & x \in [0,1]\,, t \in [0,T]\,, \\ u(x,0) = u_0(x) & x \in [0,1]\,, \\ u(0,t) = a(t) & t \in (0,T]\,, \\ u(1,t) = b(t) & t \in (0,T] \end{cases}$$

Find the discretization error (or “truncation error”) of the scheme

$$\frac{U_n^{j+1} - U_n^j}{\Delta t} = \frac{\left(U_{n+1}^j - U_n^j\right) p_{j+1/2} - \left(U_n^j - U_{n-1}^j\right) p_{j-1/2}}{(\Delta x)^2}$$

where $p_j = p(x_j) = p(j\Delta x)$