2005S M172 Exam 2 Preparation

The second midterm and the first midterm makeup are to be held during the final exam period, Monday, June 6, 11:30am-2:30pm, in the regular class room, HSS 1138.

The second midterm exam is a "one hour" exam covering the second half of the class, *i.e.*, solving hyperbolic and elliptic PDE by finite difference methods, and solving elliptic PDE by the finite element method. You have the *option* of retaking the first midterm. The retake is also a one hour exam, covering general PDEs, and specific methods for solving parabolic PDEs, stability, etc. If you elect to try the retake, you will have only the 3 hours to complete both exams. If you do not wish to take the retake, you will have the full 3 hours to take exam 2.

You will not be penalized for trying the retake–I will consider the max of the two scores to be your first exam score. However, you can only improve your first exam score up to a 80. This means that there is no reason to try the retake if you scored better than 80on the first exam, since you cannot improve your grade by retaking. In symbols, the grading scheme is as follows:

$$ex_1' \leftarrow ex_1 \vee (retake \wedge 80) = \max\{ex_1, \min\{retake, 80\}\}.$$

The following formula will be provided on both exams:

$$1 - \cos\theta = 2\sin^2(\theta/2)$$

Everything else must be committed to memory.

This prep sheet only covers the second midterm. For the first midterm retake, see your first midterm, and the first midterm prep sheet.

1. State the CFL condition. Is this condition sufficient for stability and convergence? Consider the transport equation:

$$\begin{cases} u_t + au_x = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases}$$

What is the Domain of Dependance of this PDE, at a given point (x_i, t_j)?
Give the FTFX scheme for this problem. What is the Domain of Dependance for this scheme? Under what conditions are the CFL condition satisfied for this scheme? By Fourier analysis, under what conditions is this scheme stable?

2. Suppose that the FTFX scheme is applied to the transport equation, and the mesh sizes are chosen such that $a\Delta t = -\Delta x$. Argue that the scheme is exact in this case, *i.e.*, that there is no error.

3. Let $\{\boldsymbol{v_i}\}_{i=1}^n$, $\{\lambda_i\}_{i=1}^n$ be the eigenvectors, -values of matrix A. Suppose $\lambda_1 > 1$, and $|\lambda_i| < 1$ for $i = 2, 3, \ldots, n$. Let $\boldsymbol{U^0}$ be some random vector. Make an argument that

$$\boldsymbol{U^n} = \mathsf{A}^n \boldsymbol{U^0}$$

tends towards $\lambda_1^n \boldsymbol{v_1}$.

Now suppose you are considering a "conservative" finite difference scheme for the transport equation; this scheme is explicit, and counteracts damping by unitizing at each iterate. That is, $\boldsymbol{U^0}$ is a unit vector that corresponds to the initial value data of the problem, and at each time step you make the approximation:

$$\boldsymbol{V^{j+1}} = \mathsf{A}\boldsymbol{U^j}, \qquad \boldsymbol{U^{j+1}} = \frac{\boldsymbol{V^{j+1}}}{\left\|\boldsymbol{V^{j+1}}\right\|}$$

Make some guesses about what happens to $\boldsymbol{U^j}$ as $j \to \infty$.

4. The *total variation* of a bi-infinite sequence $\{U_i\}_{i=-\infty}^{\infty}$ is the sum

$$\sum_i |U_i - U_{i-1}|$$

A differencing scheme is said to be *total variation diminishing* if

$$\sum_i \left|U_i^{j+1} - U_{i-1}^{j+1}\right| \leq \sum_i \left|U_i^j - U_{i-1}^j\right|$$

Show that the FTFX scheme for the advection equation has this property when it is stable, *i.e.*, when $-\Delta x \leq a\Delta t \leq 0$. Show that the Lax Wendroff scheme does not have this property for any Δt.

5. Give a discretization of the one dimensional elliptic "P"DE

$$\begin{cases} u_{xx} = f & x \in (0,1)\,, \\ u(0) = u_0 & \\ u(1) = u_1 & \end{cases}$$

Divide $[0,1]$ into m pieces. By Taylor's theorem, what is the size of the discretization error?

6. Consider the Dirichlet problem for *Poisson's Equation*, *i.e.*,

$$\begin{cases} \nabla^2 u = u_{xx} + u_{yy} = f & (x,y) \in \mathcal{D}, \\ u(x,y) = \phi(x,y) & (x,y) \in \partial\mathcal{D}. \end{cases}$$

Let v have the property that $\nabla^2 v$ is defined in $\mathcal{D}$, and that $v(x,y) = \phi(x,y)$ on $\partial\mathcal{D}$. Show that to solve the original problem, it suffices to find w such that

$$\begin{cases} \nabla^2 w = g =_{\text{df}} f - \nabla^2 v & (x,y) \in \mathcal{D}, \\ w(x,y) = 0 & (x,y) \in \partial\mathcal{D}. \end{cases}$$

Then to let $u = w + v$.
(This shows that when considering the Dirichlet problem for Poisson's Equation we can always assume that $\phi(x,y) = 0$.)

7. Let A be a symmetric, positive definite, $n \times n$ matrix, with $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ the discretization of an elliptic PDE. Consider the iterative method

$$\mathsf{Q}\boldsymbol{x}^{(k+1)} = (\mathsf{Q} - \mathsf{A})\,\boldsymbol{x}^{(k)} + \boldsymbol{b}.$$

Identify the Q such that this method is

1. Richardson's Method.
2. Jacobi iteration.
3. Gauss Seidel iteration.

Write out, in pseudo code, the Jacobi method, and the Gauss Seidel method.

8. What does it mean for $\boldsymbol{v}, \lambda$ to be eigenvector and eigenvalue of matrix A?

1. Suppose $\{\lambda_i\}_{i=1}^{n}$ are the eigenvalues of matrix A. What are the eigenvalues of A^{-1}.
2. Suppose $\{\lambda_i\}_{i=1}^{n}$ are the eigenvalues of matrix A. What are the eigenvalues of $\mathsf{B} = p(\mathsf{A})$, where $p(x)$ is a polynomial?
3. Suppose that the condition number of A is 1, *i.e.*, that A has a single distinct real nonzero eigenvalue, call it λ. Show that every nonzero vector is an eigenvector for A. Argue that A is $\lambda\mathsf{I}$.

9. Let $\boldsymbol{v}$ be an n-dimensional vector, and let A be an $n \times n$ matrix.

1. Define the norm $\|\boldsymbol{v}\|_2$.
2. Define the norm $\|\mathsf{A}\|_2$.
3. Show that $\|\mathsf{A}\boldsymbol{v}\|_2 \leq \|\mathsf{A}\|_2 \|\boldsymbol{v}\|_2$.
4. Show that $\|\mathsf{A}\|_2$ equals $\rho(\mathsf{A})$, the spectral radius of A. (*Note:* the spectral radius of matrix A, written $\rho(\mathsf{A})$, is its maximum eigenvalue in absolute value, *i.e.*, $\max_{\lambda_j} |\lambda_j|$ over all eigenvalues λ_j of A.)
5. Show that $\left\|\mathsf{A}^{-1}\right\|_2$ equals $(1/|\lambda_{min}|)$, where λ_{min} is the smallest, in absolute value, eigenvalue of A.
6. Define the condition number of a matrix, $\kappa(\mathsf{A})$. Show that $\kappa(\mathsf{A}) = \|\mathsf{A}\|_2 \left\|\mathsf{A}^{-1}\right\|_2$.
7. Suppose there is some $\mu > 0$ such that, for a given A,
$$\|\mathsf{A}\boldsymbol{v}\|_2 \geq \mu \|\boldsymbol{v}\|_2 ,$$
for all vectors $\boldsymbol{v}$.
   (a) Show that $\mu \leq \|\mathsf{A}\|_2$. (Should be very simple.)
   (b) Show that A is nonsingular. (*Recall:* A is singular if there is some $\boldsymbol{x} \neq \boldsymbol{0}$ such that $\mathsf{A}\boldsymbol{x} = \boldsymbol{0}$.)
   (c) Show that $\left\|\mathsf{A}^{-1}\right\|_2 \leq (1/\mu)$.
8. If A is singular, is it necessarily the case that $\|\mathsf{A}\|_2 = 0$?
9. If $\|\mathsf{A}\|_2 \geq \mu > 0$ does it follow that A is nonsingular?

10. Let A be a symmetric $n \times n$ matrix. We wish to solve the problem

$$\mathsf{A}\boldsymbol{x} = \boldsymbol{b}.$$

by using the iteration

$$\boldsymbol{x}^{(k+1)} = (\mathsf{I} - \mathsf{A})\,\boldsymbol{x}^{(k)} + \boldsymbol{b}.$$

Let $\boldsymbol{e}^{(k)} = \boldsymbol{x}^{(k)} - \boldsymbol{x}$, where $\boldsymbol{x}$ solves the linear system.

1. Suppose $0.25 \leq \left\|\boldsymbol{e}^{(k+1)}\right\|_2 / \left\|\boldsymbol{e}^{(k)}\right\|_2 \leq 0.5$. What can you say about the eigenvalues of $\mathsf{B} = \mathsf{I} - \mathsf{A}$? Then what can you say about the eigenvalues of A?
2. What is the limit, as $k \to \infty$, of $\boldsymbol{x}^{(k)}$? (*Hint:* what is the limit of $\boldsymbol{e}^{(k)}$?)
3. Suppose we start with $\boldsymbol{x}^{(0)} = \boldsymbol{0}$. How many iterations are needed to obtain an approximate solution with accuracy of 1×10^{-6}?

4. Can you pick an ω such that the "extrapolated" iteration

$$\boldsymbol{x}^{(k+1)} = (\mathsf{I} - \omega \mathsf{A})\, \boldsymbol{x}^{(k)} + \omega \boldsymbol{b}$$

gives better convergence rates? For your ω what is the upper bound on $\left\|\boldsymbol{e}^{(k+1)}\right\|_2 / \left\|\boldsymbol{e}^{(k)}\right\|_2$?

11. Let A be a symmetric positive definite $n \times n$ matrix with n distinct eigenvalues. Letting $\boldsymbol{y}^{(0)} = \boldsymbol{b} / \|\boldsymbol{b}\|_2$, consider the iteration

$$\boldsymbol{y}^{(k+1)} = \frac{\mathsf{A}\boldsymbol{y}^{(k)}}{\left\|\mathsf{A}\boldsymbol{y}^{(k)}\right\|_2}.$$

1. What is $\left\|\boldsymbol{y}^{(k)}\right\|_2$?
2. Show that $\boldsymbol{y}^{(k)} = \mathsf{A}^k \boldsymbol{b} / \left\|\mathsf{A}^k \boldsymbol{b}\right\|_2$.
3. Show that as $k \to \infty$, $\boldsymbol{y}^{(k)}$ converges to the (normalized) eigenvector associated with the largest eigenvalue of A.

12. Let A be a symmetric, positive definite, $n \times n$ matrix, with $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ the discretization of an elliptic PDE. Consider the iterative method

$$\mathsf{Q}\boldsymbol{x}^{(k+1)} = (\mathsf{Q} - \omega \mathsf{A})\, \boldsymbol{x}^{(k)} + \omega \boldsymbol{b}.$$

Show that

$$\boldsymbol{e}^{(k+1)} = \left(\mathsf{I} - \omega \mathsf{Q}^{-1} \mathsf{A}\right) \boldsymbol{e}^{(k)}.$$

1. Show that a necessary and sufficient condition for convergence of this method is that the spectral radius of $\mathsf{I} - \omega \mathsf{Q}^{-1}\mathsf{A}$ be less than 1.
2. Supposing the eigenvalues of $\mathsf{Q}^{-1}\mathsf{A}$ are in $[\alpha, \beta]$ with $0 < \alpha$, where are the eigenvalues of $\mathsf{I} - \omega \mathsf{Q}^{-1}\mathsf{A}$? Pick ω to minimize the spectral radius of $\mathsf{I} - \omega \mathsf{Q}^{-1}\mathsf{A}$.

13. Let A be a symmetric, positive definite, $n \times n$ matrix.

1. Define the A energy norm, $\|\boldsymbol{v}\|_{\mathsf{A}}$.
2. What does it mean for a set of vectors, $\{\boldsymbol{v}_i\}_{i=1}^n$ to be A-orthonormal?

14. Let A be a symmetric, positive definite, $n \times n$ matrix, with $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ the discretization of an elliptic PDE.

1. Show that solving this linear equation is equivalent to minimizing

$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top \mathsf{A} \boldsymbol{x} - \boldsymbol{x}^\top \boldsymbol{b}.$$

2. For given $\boldsymbol{x}$, $\boldsymbol{v}$, define $g(t) = f(\boldsymbol{x} + t\boldsymbol{v})$. Using freshman calculus, find the t that minimizes $g(t)$. (*Note:* you will use the fact that A is positive definite.)
3. Describe, in pseudo code, the *steepest descent* iterative method for minimizing this quadratic form.
4. Presupposing the existence of a collection of A-orthonormal vectors, $\{\boldsymbol{v}_i\}_{i=1}^n$, describe, in pseudo code, the *conjugate directions* iterative method for minimizing this quadratic form.

15. Consider the one dimensional elliptic "P"DE

$$\begin{cases} u'' = 42x^5 + 4 \quad x \in (0,1), \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$

You will consider the weak solution to this problem. Let

$$\mathcal{H} = \left\{ v\colon \mathbb{R} \to \mathbb{R} \,\middle|\, \int_0^1 |v'|^2 \,\mathrm{d}x < \infty,\ v(0) = v(1) = 0 \right\}$$

1. Pick any $v \in \mathcal{H}$, multiply it by both sides of $u'' = 42x^5 + 4$ and integrate over $[0,1]$. Use integration by parts and the boundary values of v to get something like

$$\int_0^1 u'v' \,\mathrm{d}x = \int_0^1 ? \,\mathrm{d}x \tag{1}$$

2. Let $\mathcal{S}_H \subseteq \mathcal{H}$ be all polynomial functions of $\mathcal{H}$ of degree no greater than 3.
   a) Argue that $\mathcal{S}_H$ is a vector space of degree 2.
   b) Argue that $\phi_1(x) = x^3 - x$, and $\phi_2(x) = x^2 - x$ are in $\mathcal{S}_H$.
   c) Prove that $\mathcal{S}_H = \operatorname{span}\{\phi_1, \phi_2\}$.
3. Under the Galerkin formulation, you want to find $u_h^* \in \mathcal{S}_H$ such that equation 1 holds for all $v \in \mathcal{S}_H$.
   a) Evaluate the following:

$$\int_0^1 x^5 \phi_1(x) \,\mathrm{d}x \qquad \int_0^1 x^5 \phi_2(x) \,\mathrm{d}x$$

$$\int_0^1 \phi_1(x) \,\mathrm{d}x \qquad \int_0^1 \phi_2(x) \,\mathrm{d}x$$

$$\int_0^1 \phi_1'(x)\phi_1'(x) \,\mathrm{d}x \qquad \int_0^1 \phi_2'(x)\phi_2'(x) \,\mathrm{d}x \qquad \int_0^1 \phi_1'(x)\phi_2'(x) \,\mathrm{d}x$$

   b) Let $u_h^* = u_1\phi_1 + u_2\phi_2$. Setting $v = \phi_1$, and $v = \phi_2$, use your integral evaluations above to rewrite your integral as a linear system:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

   c) Find u_1, u_2.
   d) Compare u_h^* with the actual answer $u(x) = x^7 + 2x^2 - 3x$.

16. Consider the one dimensional elliptic "P"DE

$$\begin{cases} (xu')' + u = x & x \in (0,1), \\ u'(0) = 1 \\ u(1) = 0 \end{cases}$$

You will consider the weak solution to this problem. Let

$$\mathcal{H} = \left\{ v\colon \mathbb{R} \to \mathbb{R} \,\middle|\, \int_0^1 |v''|^2 \,\mathrm{d}x < \infty,\ v(1) = 0 \right\}$$

1. Multiply the PDE by $v \in \mathcal{H}$, and take the integral over $[0, 1]$.
2. Use integration by parts to expand $(xu')'\, v$. Simplify this expression assuming that $u, v \in \mathcal{H}$.
3. Simplify your equation to get something like

$$\int_0^1 (? - ?)\, \mathrm{d}x = \int_0^1 xv\, \mathrm{d}x.$$

4. Let $\mathcal{S}_H \subseteq \mathcal{H}$ be the subset of all polynomial functions of $\mathcal{H}$ of degree no greater than 2.
   1. Argue that $\phi_1(x) = x - 1$, and $\phi_2(x) = x^2 - 1$ are in $\mathcal{S}_H$.
   2. Prove that $\mathcal{S}_H = \operatorname{span}\{\phi_1, \phi_2\}$. (*Hint:* A typical element of $\mathcal{S}_H$ is of the form $v(x) = ax^2 + bx + c$. Use this and the fact that $v(1) = 0$ to express v as the linear combination of ϕ_1, ϕ_2.)
   3. Evaluate the following:

$$\int_0^1 x\phi_1(x)\, \mathrm{d}x \qquad \int_0^1 x\phi_2(x)\, \mathrm{d}x$$

$$\int_0^1 \phi_1(x)\phi_1(x)\, \mathrm{d}x \qquad \int_0^1 \phi_2(x)\phi_2(x)\, \mathrm{d}x \qquad \int_0^1 \phi_1(x)\phi_2(x)\, \mathrm{d}x$$

$$\int_0^1 x\phi_1'(x)\phi_1'(x)\, \mathrm{d}x \qquad \int_0^1 x\phi_2'(x)\phi_2'(x)\, \mathrm{d}x \qquad \int_0^1 x\phi_1'(x)\phi_2'(x)\, \mathrm{d}x$$

5. Let $u = u_1\phi_1 + u_2\phi_2$. Setting $v = \phi_1$, and $v = \phi_2$, use your integral evaluations above to rewrite your integral as a linear system:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

6. Conclude that $u_1 = 1, u_2 = 0$, *i.e.*, that the best approximate weak solution to the PDE in the space $\mathcal{S}_H$ is the function $\phi_1(x) = x - 1$.
7. Verify that $u = \phi_1$ actually solves the given PDE.