2005S M172 Homework 1

This homework is due April 11 in discussion section.

1. Classify each of the following PDEs as elliptic, hyperbolic, parabolic:

1. $u_{xx} + 2u_{xy} + u_{yy} + u_x = 1$.
2. $u_{xx} + u_{xy} + u_{yy} + u = x^2$.
3. $u_{xx} + u_{xy} - u_{yy} + u_y = y - x$.
4. $u_{xx} + 2x^2 u_{xy} - u_{yy} + u = 0$.
5. $x^2 u_{xx} + y^2 u_{yy} + x u_x + y u_y = 0$.

2. Use the maximum principle of Harmonic functions to prove that the Dirichlet problem for Laplace's equation depends continuously on the input data. That is, suppose u_1 solves the (Dirichlet) problem

$$\begin{cases} \Delta u_1(\boldsymbol{x}) = f(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \Omega \\ u_1(\boldsymbol{x}) = \phi_1(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \partial\Omega \end{cases}$$

and u_2 satisfies

$$\begin{cases} \Delta u_2(\boldsymbol{x}) = f(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \Omega \\ u_2(\boldsymbol{x}) = \phi_2(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \partial\Omega \end{cases}$$

where $|\phi_1(\boldsymbol{x}) - \phi_2(\boldsymbol{x})| \leq \epsilon$ for $\boldsymbol{x} \in \partial\Omega$. Let $u_3 = u_1 - u_2$.

1. Show that u_3 satisfies the (Homogeneous) Dirichlet problem:

$$\begin{cases} \Delta u_3(\boldsymbol{x}) = 0 & \text{for } \boldsymbol{x} \in \Omega \\ u_3(\boldsymbol{x}) = \phi_3(\boldsymbol{x}) = \phi_1(\boldsymbol{x}) - \phi_2(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \partial\Omega \end{cases}$$

2. Argue that

$$\max_{\boldsymbol{x} \in \partial\Omega} \phi_3(\boldsymbol{x}) \leq \epsilon \quad \text{and} \quad \min_{\boldsymbol{x} \in \partial\Omega} \phi_3(\boldsymbol{x}) \geq -\epsilon$$

3. Use the maximum/minimum principle to show that $|u_1(\boldsymbol{x}) - u_2(\boldsymbol{x})| = |u_3(\boldsymbol{x})| \leq \epsilon$ for $\boldsymbol{x} \in \Omega$.

3. Consider the (trivial) wave equation with initial value data:

$$\begin{cases} u_x(x,t) - \frac{1}{c} u_t(x,t) = 0 & \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = \phi(x) & \text{for } x \in \mathbb{R} \end{cases}$$

Consider the change of variables $(\xi, \eta) = T(x,t) = (x + ct, x - ct)$. Note this is a linear transform with inverse $(x,t) = T^{-1}(\xi,\eta) = ((\xi+\eta)/2, (\xi-\eta)/2c)$ Suppose u is a solution to the above problem, and let $\tilde{u} = u \circ T^{-1}$, that is let $\tilde{u}(\xi,\eta) = u\left((\xi+\eta)/2, (\xi-\eta)/2c\right)$.

1. Use the chain rule to show that

$$\frac{\partial \tilde{u}}{\partial \xi} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{2c}\frac{\partial u}{\partial t}, \quad \text{and} \quad \frac{\partial \tilde{u}}{\partial \eta} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2c}\frac{\partial u}{\partial t}$$

2. It should immediately follow that

$$\frac{\partial \tilde{u}}{\partial \eta} = 0,$$

and therefore $\tilde{u}(\xi, \eta) = w(\xi) = w(x + xt)$ for some function w.

3. Prove that w must be ϕ, and therefore that $u(x,t) = \phi(x + ct)$ uniquely solves this problem.

4. You have just shown that this problem has a unique solution, now prove that the solution depends continuously on the input data ϕ. That is, suppose u_1 uniquely solves the problem for initial data ϕ_1, and u_2 solves it uniquely for ϕ_2, and furthermore assume that

$$\max_{x \in \mathbb{R}} |\phi_1(x) - \phi_2(x)| \leq \epsilon$$

then prove that

$$\max_{x \in \mathbb{R},\, t \geq 0} |u_1(x,t) - u_2(x,t)| \leq \epsilon$$

(*Hint:* this is supposed to be really easy.)

5. What does the solution $u(x,t)$ "look like"? *i.e.*, how should this solution be physically interpreted.

4. Do problems 1.1,1.4,1.5,1.6,1.10,1.11 of the notes [1].

References

[1] Randolph E. Bank, Peter Rentrop and Donald R. Smith. Numerical treatment of partial differential equations, 1995. Course Notes for UCSD Math M172.