2005S M172 Homework 2

This homework is due April 20 in class.

1. Approximate the solution to the elliptic PDE:

$$\begin{cases} u_{xx} + u_{yy} = \sin(\pi x)\cos(\pi y) & \text{on} \quad [0,2] \times [0,3] \\ u(0,y) = y & \text{for} \quad y \in [0,3] \\ u(2,y) = y + 1 & \text{for} \quad y \in [0,3] \\ u(x,0) = 3x & \text{for} \quad x \in [0,2] \\ u(x,3) = x^2 - \frac{3}{2}x + 3 & \text{for} \quad x \in [0,2] \end{cases}$$

Use finite differences. Let $\Delta x = \Delta y = 1$. You should have 12 degrees of freedom, 10 of which are determined by the boundary condition. This reduces the problem to two equations in two unknowns.

2. Do problem 2.1 of the notes [1].

3. Let n be a largeish integer, let k, l be integers with $1 \leq k, l \leq n$. Define

$$\theta_k = \frac{k\pi}{n+1} \quad \text{and} \quad \theta_l = \frac{l\pi}{n+1}$$

Recall that $\lambda_k = 2 - 2\cos\theta_k$, and $\lambda_l = 2 - 2\cos\theta_l$ are eigenvalues of

$$\mathsf{B} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix}$$

with associated eigenvectors $\boldsymbol{v_k}$, $\boldsymbol{v_l}$ where

$$\boldsymbol{v_k} = \begin{bmatrix} \sin(1\theta_k) \\ \sin(2\theta_k) \\ \sin(3\theta_k) \\ \vdots \\ \sin(n\theta_k) \end{bmatrix} \quad \text{and} \quad \boldsymbol{v_l} = \begin{bmatrix} \sin(1\theta_l) \\ \sin(2\theta_l) \\ \sin(3\theta_l) \\ \vdots \\ \sin(n\theta_l) \end{bmatrix}$$

Your job is to show that $\boldsymbol{v_k}$ and $\boldsymbol{v_l}$ are orthogonal. This would then imply that the collection $\{\boldsymbol{v_i}\}_{i=1}^{n}$ form a basis for $\mathbb{R}^n$. To do this, assume $k \neq l$ and show $\boldsymbol{v_k} \cdot \boldsymbol{v_l} = 0$.

1. (Easy) First show that

$$\boldsymbol{v_k} \cdot \boldsymbol{v_l} = \sum_{j=1}^{n} \sin(j\theta_k)\sin(j\theta_l)$$

2. Now use trig identities to prove that

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

3. Use the previous two parts and the fact that $\cos(x) = \Re\left(e^{ix}\right)$ to show that

$$\boldsymbol{v_k} \cdot \boldsymbol{v_l} = \frac{1}{2}\sum_{j=1}^{n} \Re\left(e^{ij(\theta_k-\theta_l)}\right) - \Re\left(e^{ij(\theta_k+\theta_l)}\right) = \frac{1}{2}\Re\left[\sum_{j=1}^{n} e^{ij(\theta_k-\theta_l)} - \sum_{j=1}^{n} e^{ij(\theta_k+\theta_l)}\right]$$

where $\Re$ denotes the real part of a complex number: $\Re(a+bi) = a$.

4. (Easy) Show that, if $z \neq 1$ then

$$\sum_{j=1}^{n} z^j = \begin{cases} \frac{z^{n+1}-z}{z-1} & \text{if } z \neq 1 \\ n & \text{if } z = 1 \end{cases}$$

(Gauss reputedly proved this at the age of ten.)

5. Let $z_- = e^{i(\theta_k-\theta_l)}$ and $z_+ = e^{i(\theta_k+\theta_l)} = z_- e^{i2\theta_l}$. Show that if $k-l$ is even and nonzero, then $z_-^{n+1} = z_+^{n+1} = 1$, and so $\sum_{j=1}^{n} z_-^j = \sum_{j=1}^{n} z_+^j = -1$. Argue that $\boldsymbol{v_k} \cdot \boldsymbol{v_l} = 0$ in this case.

6. (Harder) If $k-l$ is odd, show that $z_-^{n+1} = z_+^{n+1} = -1$, and thus

$$\boldsymbol{v_k} \cdot \boldsymbol{v_l} = \frac{1}{2}\Re\left[\frac{z_+ + 1}{z_+ - 1} - \frac{z_- + 1}{z_- - 1}\right]$$

Argue that

$$\Re\left[\frac{z_+ + 1}{z_+ - 1}\right] = \Re\left[\frac{z_- + 1}{z_- - 1}\right] = 0.$$

Do this by rationalizing denominators, *i.e.*, prove that

$$\Re\left[\frac{(z_+ + 1)(\overline{z_+} - 1)}{(z_+ - 1)(\overline{z_+} - 1)}\right] = 0,$$

and similarly for z_-.

7. (Bonus) Show that $\boldsymbol{v_k} \cdot \boldsymbol{v_k} = (n+1)/2$.

8. Consider yourself a Fourier Analysis Wizard.

References

[1] Randolph E. Bank, Peter Rentrop and Donald R. Smith. Numerical treatment of partial differential equations, 1995. Course Notes for UCSD Math M172.