

## 2005S M172 Homework 4

This homework is due May 9 in discussion section.

1. Do problem 3.5 and 3.6 of the notes [1]. These problems deal with any implicitly defined scheme for the heat equation. Any implicit scheme (including Crank Nicolson, and “the” implicit scheme) solve for a given time slice by solving an equation of the form:

$$\mathbf{B}\vec{U}^{j+1} = \vec{G}^j,$$

where  $\vec{G}^j$  relies on  $\vec{U}^j$  and on  $\vec{F}^j$  and the boundary values.

The matrix  $\mathbf{B}$  is usually tridiagonal and Toeplitz. Questions 3.5 and 3.6 ask how you would efficiently implement an implicit scheme.

2. (**Morton & Mayers #2.6**) Define the operator “ $\delta_x^2$ ” as follows:

$$\delta_x^2 U_n^j =_{\text{df}} U_{n+1}^j - 2U_n^j + U_{n-1}^j.$$

1. Show that the scheme

$$U_n^{j+1} - U_n^{j-1} = \frac{s}{3} [\delta_x^2 U_n^{j+1} + \delta_x^2 U_n^j + \delta_x^2 U_n^{j-1}]$$

is stable for all values of  $s = \Delta t / (\Delta x)^2$ .

2. Show that the scheme

$$U_n^{j+1} - U_n^{j-1} = \frac{s}{6} [\delta_x^2 U_n^{j+1} + 4\delta_x^2 U_n^j + \delta_x^2 U_n^{j-1}]$$

is unstable for all values of  $s$ .

You should do these via Fourier Analysis. For both parts you may use the **Amazing Fact**: Equation  $z^2 + bz + c = 0$  with real coefficients  $b, c$ , has (complex) roots on or within the unit circle if and only if  $|c| \leq 1$  and  $|b| \leq 1 + c$ .

3. (**extra credit**) Prove the **Amazing Fact** above.

## References

- [1] Randolph E. Bank, Peter Rentrop and Donald R. Smith. Numerical treatment of partial differential equations, 1995. Course Notes for UCSD Math M172.