2005S M172 Homework 6. This homework is due Monday, May 23.

1. Do problem 5.1 of the notes [1].

2. Consider the Jacobi iterative method for solving the system of n equations

$$\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$$

It works as follows: let $\boldsymbol{x}^{(0)}$ be some starting guess of the solution. Let ω be some preselected weighting. Let Q be the diagonal matrix whose diagonal is the same as that of A. Then implicitly define $\boldsymbol{x}^{(k+1)}$ as follows:

$$\mathsf{Q}\boldsymbol{x}^{(k+1)} = (\mathsf{Q} - \omega\mathsf{A})\,\boldsymbol{x}^{(k)} + \omega\boldsymbol{b}$$

1. Show that the Jacobi method can be implemented as follows:
   - Set $\boldsymbol{y} = \boldsymbol{x}^{(0)}$.
   - For $k = 1, 2, \ldots, K$

     Set $\boldsymbol{x} = \boldsymbol{y}$. (We will compute $\boldsymbol{x}^{(k)}$ as $\boldsymbol{y}$, and let $\boldsymbol{x}$ hold $\boldsymbol{x}^{(k-1)}$.)

     For $j = 1, 2, \ldots, n$,

     $y_j \leftarrow x_j + \frac{\omega}{a_{jj}}\left(b_j - \sum_{i=1}^{n} a_{ji}x_i\right)$.
   - Output $\boldsymbol{y}$ as $\boldsymbol{x}^{(K)}$.

2. Suppose your job is to solve $\mathsf{B_h}\boldsymbol{x} = \boldsymbol{b}$, where $\mathsf{B_h}$ is defined to be $(1/h^2)\mathsf{B}$, and

$$\mathsf{B} = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{bmatrix}$$

   Recall that the $n \times n$ matrix B has eigenvalue $-4\sin^2(\theta_k/2)$ with eigenvector $\boldsymbol{v_k} = [\sin(1\theta_k), \sin(2\theta_k), \ldots, \sin(n\theta_k)]^\top$, where $\theta_k = \frac{k\pi}{n+1}$ for any k, $1 \le k \le n$.
   Taking Q to be the diagonal of $\mathsf{B_h}$, show that

$$\mathsf{I} - \omega\mathsf{Q}^{-1}\mathsf{B_h} = \mathsf{I} + (\omega/2)\,\mathsf{B}$$

3. The requirement

$$\left\|\mathsf{I} - \omega\mathsf{Q}^{-1}\mathsf{B_h}\right\|_2 < 1$$

   is sufficient to prove convergence for the Jacobi Method. Show that any $\omega \in (0, 1)$ gives this inequality. (Recall that $\|\mathsf{C}\|_2$ is the largest, in absolute value, eigenvalue of C.)

4. Implement the Jacobi Method in your favorite programming language. Here's a hint if you're going to use Matlab:

```
function y = jacobi(A,b,omega,x0,K)
% run jacobi to get approximate solution to Ax=b,
% starting with x0, using weighting omega and returning x^(k)
% written for clarity, not for speed...

% initialization
[n,m]   = size(A);
...

for k = 1:K
  x = y;
  for j = 1:n
    y(j) = x(j) + (omega/a(j,j)) * (b(j) - a(j,:) * x);
  end
end
...
```

PLEASE WRITE YOUR OWN CODE. NO JOKE.

5. Test your code on the following problem: let $n = 40$ (or bigger), let

$$\boldsymbol{x}_{sol} = [\sin(1\theta_n), \sin(2\theta_n), \sin(3\theta_n), \ldots, \sin(n\theta_n)]^\top$$

and *define* $\boldsymbol{b}$ by $\boldsymbol{b} = \mathsf{B_h}\boldsymbol{x}_{sol} = -(4/h^2)\sin^2(\theta_n/2)\boldsymbol{x}_{sol}$. Use your code to attempt to solve $\mathsf{B_h}\boldsymbol{x} = \boldsymbol{b}$. Let your starting vector $\boldsymbol{x}^{(0)}$ be a random vector with elements in $[-1, 1]$. Let $\omega = 2/3$.
Because you know $\boldsymbol{x}_{sol}$, you can easily find the error of your approximate: $\boldsymbol{e}^{(k)} = \boldsymbol{x}^{(k)} - \boldsymbol{x}_{sol}$. Plot the vector $\boldsymbol{e}^{(k)}$ for $k = 0, 1, 2, 5, 10, 20, 50$. Can you see why this iterative process is called a "smoother"?

6. *(Bonus)* Because $\boldsymbol{x}_{sol}$ is an eigenvector of the system we are trying to solve, it seems like we could exploit this information to choose a really good ω. Is this true? Can you find an ω such that your code converges really quickly?

References

[1] Randolph E. Bank, Peter Rentrop and Donald R. Smith. Numerical treatment of partial differential equations, 1995. Course Notes for UCSD Math M172.