2005S M172 Homework 7. This homework is due Friday, June 3.

1. Let $(\, , \,)$ be a positive, symmetric, bilinear form over the vector space $\mathcal{V}$:
- for $u, v, w \in \mathcal{V}$, $(u + v, w) = (u, w) + (v, w)$, and $(u, v + w) = (u, v) + (u, w)$,
- for $u, v \in \mathcal{V}$, and scalar α, $(\alpha u, v) = \alpha (u, v) = (u, \alpha v)$,
- for $u, v \in \mathcal{V}$, $(u, v) = (v, u)$,
- for $u \in \mathcal{V}$, $(u, u) \geq 0$, with equality if u is the zero vector.

You will prove the Cauchy Schwarz Inequality:

$$(u, v) \leq \sqrt{(u, u)} \sqrt{(v, v)},$$

also written as $(u, v) \leq \|u\| \, \|v\|$, where $\|w\| = \sqrt{(w, w)}$.

1. Pick vectors u, v and claim, trivially, that $(u + tv, u + tv) \geq 0$ where t is a scalar. Expand $(u + tv, u + tv)$ as a quadratic in t, *i.e.*, as $\alpha t^2 + 2\beta t + \gamma$.
2. Because $\alpha t^2 + 2\beta t + \gamma \geq 0$, show that if $\alpha = 0$, then $\beta = 0$, which proves the Cauchy Schwarz Inequality in the degenerate case when $(v, v) = 0$.
3. Now suppose $\alpha \neq 0$. How many roots can this quadratic have? What does this say about the discriminant $4\beta^2 - 4\alpha\gamma$? Use this to show the Cauchy Schwarz Inequality.

2. Let Ω be an open bounded subset of $\mathbb{R}^d$. Define the inner product $(\, , \,)$ as follows:

$$(\phi, \psi) = \int_{\Omega} \phi(x) \psi(x) \, \mathrm{d}x,$$

for functions $\phi, \psi \colon \Omega \to \mathbb{R}$.

1. Show that $(\, , \,)$ is a symmetric bilinear form. And that $(u, u) \geq 0$, with equality holding if u is the constant zero function.
2. Show that there are functions u such that $(u, u) = 0$, but u is *not* the constant zero function. (Hint: u need not be continuous.)

We would like to use the inner product above to define a normed space, but the previous part points out a problem: We want the norm to be zero only for the zero function. This is a minor annoyance, which I once pointed out in a functional analysis class; everybody looked at me like I was terribly naïve, and that I should have learned about this 'on the streets' like everybody else. There is a simple fix you will explore here.

3. Show that the relation $\equiv$ defined by

$$u \equiv v \Leftrightarrow \int_{\Omega} (u(x) - v(x))^2 \, \mathrm{d}x = 0,$$

is an equivalence relation on the set of functions from Ω to $\mathbb{R}$.

4. Use the Cauchy Schwarz Inequality to show that if $u \equiv v$, then $(u, w) = (v, w)$ for all w, and thus we *could* construct a well defined inner product $(\,,)_\equiv$ over $\{\phi\colon \Omega \to \mathbb{R}\} / \equiv$ by letting

$$([u]_\equiv, [v]_\equiv)_\equiv = (u, v),$$

where $[u]_\equiv$ is the set of functions equivalent to u under the relation $\equiv$.

5. Show that

$$([u]_\equiv, [u]_\equiv)_\equiv = 0 \Leftrightarrow [u]_\equiv = [0]_\equiv$$

In reality, we never explicitly talk about this quotient space inner product, but we always assume that we are talking about functions "modulo a set of measure zero." This is just a fancy way of saying that $u \equiv v$ exactly when $u - v$ is nonzero only on a set of "measure zero." That is, we pretend u and v are the same function if $u \equiv v$.
Now define $L^2(\Omega)$ as $\{\phi\colon \Omega \to \mathbb{R} \mid \|\phi\|_2 < \infty\}$, where the norm is defined as $\|\phi\|_2 = \sqrt{(\phi, \phi)}$.

6. Show that $L^2(\Omega)$ is a vector space, *i.e.*, that it is closed under taking scalar multiples and under addition.

3. Let $\Omega = (-1, 1)$. Consider the "P"DE:

$$\begin{cases} u_{xx}(x) = f(x) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega \end{cases}$$

Let $f(x) = 60x^4 + 2$. You will consider the Ritz and Galerkin approximations of this problem for a finite dimensional subset of functions.
Let $\mathcal{H} = \left\{\phi\colon \Omega \to \mathbb{R} \mid \int_\Omega (\phi_x(x))^2 \,\mathrm{d}x < \infty, \text{ and } \phi(x) = 0 \text{ for } x \in \partial\Omega \right\}$.

1. Show that the exact solution is $u(x) = 2x^6 + x^2 - 3$.
2. Let $\mathcal{S}_H \subseteq \mathcal{H}$ be the subset of $\mathcal{H}$ consisting of quadratic, linear and constant functions. Show that $\mathcal{S}_H$ is a one dimensional vector space. (Hint: start with the assumption that for $\phi \in \mathcal{S}_H$, that $\phi(x) = ax^2 + bx + c$, then prove that $b = 0$, and $a = -c$.)
3. Find the $u_h^* \in \mathcal{S}_H$ such that

$$a(u_h^*, v) = (f, v)$$

for all $v \in \mathcal{S}_H$, where $a(\phi, \psi) = (\phi_x, \psi_x)$, and $(\xi, \eta) = \int_\Omega \xi\eta \,\mathrm{d}x$.

4. Find the $u_h^* \in \mathcal{S}_H$ such that

$$E(u_h^*) = \min_{v \in \mathcal{S}_H} E(v), \quad \text{where} \quad E(\phi) = \frac{1}{2}a(\phi, \phi) - (\phi, f)$$

Do this directly.

4. Consider the two functions $g, h\colon \mathbb{R}^n \to \mathbb{R}$ given by

$$g(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top \mathsf{A}\boldsymbol{x}, \quad h(\boldsymbol{x}) = \boldsymbol{f}^\top \boldsymbol{x}$$

where A is a symmetric matrix.

1. Show that $\nabla g(\boldsymbol{x}) = \mathsf{A}\boldsymbol{x}$.
2. Show that $\nabla h(\boldsymbol{x}) = \boldsymbol{f}$.

Thus $f(\boldsymbol{x}) = g(\boldsymbol{x}) - h(\boldsymbol{x})$ is minimized exactly when $\nabla [f - g](\boldsymbol{x}) = \boldsymbol{0}$, *i.e.*, when $\mathsf{A}\boldsymbol{x} = \boldsymbol{f}$.