2005S M20C Exam 1 Preparation

The first midterm exam is Wednesday April 20, during the class period. *You must bring a blue book to the exam.* Blue books are available from the bookstore. You also need to bring your student ID card or other form of ID (driver's license, passport, etc.) The exam covers material from §10.1, 10.2, 12.1-12.5, 13.1–13.3, 14.1.

The following formulæ will be provided on your exam:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad \boldsymbol{a} \times \boldsymbol{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Everything else must be committed to memory.

You should be prepared to answer at least the following questions:

1. What's the difference between a scalar and a vector? Describe the input and output types of common operations.

Answer: A vector comprises a direction and a scalar magnitude. A scalar can be conceived of as a vector in 1 dimension, and thus has trivial direction (*i.e.*, in the positive direction). The magnitude (or length) of a vector is a scalar. Scalar multiplication takes a scalar and a vector and produces a vector. The dot product takes two vectors and produces a scalar. The cross product takes two vectors and produces a third vector.

2. Let $P = (2, 3, 1)$, $Q = (4, -2, 5)$.

1. What is the "position vector" of P?
2. Find the distance from P to Q.
3. Find the vector from P to Q, $\boldsymbol{PQ}$.
4. Find the distance from P to the origin.
5. Find the distance from P to the xy plane.
6. Find the equation of the sphere centered at P with radius 5.
7. Find the equation of the sphere centered at Q such that P is on the sphere.

Answer:

1. The vector $\langle 2, 3, 1 \rangle$.
2. $\sqrt{45}$.
3. $\langle 2, -5, 4 \rangle$.
4. $\sqrt{14}$.
5. It is just the z coordinate, 1.
6. $(x-2)^2 + (y-3)^2 + (z-1)^2 = 25$.
7. $(x-4)^2 + (y+2)^2 + (z-5)^2 = 45$.

3. Let $\boldsymbol{a} = \langle 1, 0, 1 \rangle$, $\boldsymbol{b} = \langle 1, 3, 0 \rangle$.

1. Find $|\boldsymbol{a}|$.
2. Find a unit vector in the same direction as $\boldsymbol{a}$.
3. Find $\boldsymbol{a} + \boldsymbol{b}$.
4. Find $3\boldsymbol{a} - 2\boldsymbol{b}$.
5. Find $\boldsymbol{a} \cdot \boldsymbol{b}$.

6. Find $\boldsymbol{a} \times \boldsymbol{b}$.
7. Find a vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.
8. Find the angle subtended by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
9. Find the component of $\boldsymbol{a}$ along $\boldsymbol{b}$.
10. Find the projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.
11. Give a parametrization of the line through the point $(3, 4, 1)$ parallel to $\boldsymbol{b}$.
12. Give a parametrization of the plane through the origin that is parallel to both $\boldsymbol{a}$ and $\boldsymbol{b}$.
13. Give a parametrization of the plane containing the point $(1, 2, -1)$ that has $\boldsymbol{a}$ as a normal.
14. Find the angle subtended by $\boldsymbol{a}$ and $\boldsymbol{a} \times \boldsymbol{b}$.
15. Is $\boldsymbol{a}$ parallel to $\boldsymbol{b}$? Are they perpendicular?
16. Find a vector parallel to $\boldsymbol{a}$.

Answer:

1. $\sqrt{2}$.
2. $\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$.
3. $\langle 2, 3, 1 \rangle$.
4. $\langle 1, -6, 3 \rangle$.
5. 1.
6. $\langle -3, 1, 3 \rangle$.
7. $\langle -3, 1, 3 \rangle$.
8. $\arccos \left(\frac{1}{2\sqrt{5}} \right)$.
9. $\frac{1}{\sqrt{10}}$.
10. $\left\langle \frac{1}{10}, \frac{3}{10}, 0 \right\rangle$.
11. $\langle 3, 4, 1 \rangle + t \langle 1, 3, 0 \rangle$.
12. $-3x + y + 3z = 0$.
13. $x + z = 0$.
14. It should be $\pi/2$.
15. They are neither parallel nor perpendicular.
16. $\langle l, 0, l \rangle$, for any $l \neq 0$, is parallel to $\boldsymbol{a}$.

4. Let $\boldsymbol{u} = \langle 1, 2, -3 \rangle$, $\boldsymbol{v} = \langle -2, 4, 1 \rangle$, $\boldsymbol{w} = \langle -3, 0, 1 \rangle$.

1. Find $5\boldsymbol{u}$.
2. Find $\boldsymbol{u} \cdot \boldsymbol{u}$.
3. Find $|\boldsymbol{u}|$.
4. Find a unit length vector in the same direction as $\boldsymbol{u}$.
5. Find $|\alpha \boldsymbol{u}|$, where α is a real number.
6. Find $\boldsymbol{u} \cdot \boldsymbol{v}$.
7. Find the angle subtended by $\boldsymbol{u}$ and $\boldsymbol{v}$.

8. Find $\boldsymbol{v} \times \boldsymbol{w}$.
9. Find a vector perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
10. Find a unit length vector perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
11. Find $\boldsymbol{u} \times \boldsymbol{v}$.
12. Find $\boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})$.
13. Find $\boldsymbol{u} \times \boldsymbol{u}$.

Answer: 1. $\langle 5, 10, -15\rangle$
2. 14
3. $\sqrt{14}$
4. $(1/\sqrt{14})\boldsymbol{u} = \left\langle 1/\sqrt{14}, 2/\sqrt{14}, -3/\sqrt{14}\right\rangle$.
5. $|\alpha|\sqrt{14}$
6. $-2+8-3=3$
7. $\arccos(\boldsymbol{u}\cdot\boldsymbol{v}/|\boldsymbol{u}|\,|\boldsymbol{v}|) = \arccos\left(3/\sqrt{14}\sqrt{21}\right) \approx 80°$
8. $\boldsymbol{v} \times \boldsymbol{w} = \langle 4, -1, 12\rangle$
9. $\boldsymbol{v} \times \boldsymbol{w}$ is perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{w}$.
10. $\boldsymbol{v} \times \boldsymbol{w}/|\boldsymbol{v} \times \boldsymbol{w}| = \langle 4, -1, 12\rangle/\sqrt{161} = \left\langle 4/\sqrt{161}, -1/\sqrt{161}, 12/\sqrt{161}\right\rangle$.

5. Let $\boldsymbol{R_1}(t) = \langle 0, 3, 1\rangle + t\langle 2, 1, -1\rangle$ parametrize the line l_1. Let $\boldsymbol{R_2}(t) = \langle 1, 2, 1\rangle + t\langle 2, 3, 0\rangle$ parametrize the line l_2. Let $3x - 2y + 5z = 1$ describe a plane P.
1. Find a vector parallel to l_1. Find a vector parallel to l_2.
2. Find a vector normal to P.
3. Do the lines l_1, l_2 intersect? If so, find their point of intersection. If not, are they parallel, or are they skew?
4. Find the intersection of l_1 and P.
5. Find the intersection of l_2 and P. (*Hint:* it's a trick)
6. Find the angle that l_1 subtends with the plane P.
7. Find the distance from l_1 to the point $(2, 2, 1)$.
8. Find the distance from l_2 to P.

Answer: 1. $\langle 2, 1, -1\rangle, \langle 2, 3, 0\rangle$.
2. $\langle 3, -2, 5\rangle$.
3. They are skew.
4. $(-4, 1, 3)$.
5. They do not intersect.
6. $\pi/2 - \arccos\left(\frac{1}{2\sqrt{57}}\right) = \arcsin\left(\frac{1}{2\sqrt{57}}\right) \approx 3.8°$.
7. $\sqrt{7/2}$.
8. $3/\sqrt{38}$.

6. Let $3x - 2y + 5z = 1$ describe a plane P. Let $x + y + 3z = 4$ describe a plane Q.
1. Find a vector normal to P.
2. Find a vector normal to Q.

3. Are the planes P, Q parallel? If not, must they intersect?
4. Find the angle subtended by the two planes

Answer: 1. $\langle 3, -2, 5\rangle$.

2. $\langle 1, 1, 3\rangle$.
3. Their normals are not parallel, so they are not parallel and must intersect.
4. Find the angle subtended by the two normals. It is

$$\arccos \frac{\langle 3, -2, 5\rangle \cdot \langle 1, 1, 3\rangle}{|\langle 3, -2, 5\rangle|\,|\langle 1, 1, 3\rangle|} = \arccos \frac{16}{\sqrt{38}\sqrt{11}}$$

7. Let $\boldsymbol{u} = \langle -2, 1, 4\rangle$, and let P be the point $(0, 4, 1)$.

1. Find a vector valued function $\boldsymbol{l}(t)$ which traces out the line through P and parallel to $\boldsymbol{u}$.
2. Find the equation of the plane containing P with normal $\boldsymbol{u}$.
3. Find the parametrization, $\boldsymbol{m}(t)$, of the line line through P and perpendicular to $\boldsymbol{u}$. (*Hint:* There are many answers. One way to do this is pick some arbitrary vector $\boldsymbol{v}$ not parallel to $\boldsymbol{u}$, then find a vector perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$.)

Answer: 1. One answer is $\boldsymbol{l}(t) = \langle 0, 4, 1\rangle + t\,\langle -2, 1, 4\rangle = \langle -2t, 4 + t, 1 + 4t\rangle$.

2. Planes are of the form $ax + by + cz + d = 0$, where $\langle a, b, c\rangle$ (and any scalar multiple of it) are normal to the plane. So the answer should be of the form $-2x + y + 4z + d = 0$. Plug in P as (x, y, z) to find the value of d. You should get $-2x + y + 4z - 8 = 0$.
3. First I will let $\boldsymbol{v} = \langle 0, 0, 1\rangle$, just for simplicity. Then I compute $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v} = \langle 1, 2, 0\rangle$, which is perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$. I only care that it is perpendicular to $\boldsymbol{u}$. If I had picked the “wrong” $\boldsymbol{v}$, this cross product would have been $\mathbf{0}$, and I would have known something was wrong and picked a different $\boldsymbol{v}$. Then I do as in the first part, getting $\boldsymbol{m}(t) = \langle 0, 4, 1\rangle + t\,\langle 1, 2, 0\rangle$.

8. (§13.2) Find the limits:

1. $\lim_{t\to 0} \boldsymbol{R}(t)$, where $\boldsymbol{R}(t) = \left\langle t + 1, 5, t^2\right\rangle$.
2. $\lim_{t\to 0} \left\langle \frac{\sin t}{t}, e^{-t}, t\right\rangle$.
3. $\lim_{t\to 0} \left\langle \frac{t^2+t^4}{4t^2-t^3}, 5t + 4, \ln(1 + t)\right\rangle$.
4. $\lim_{t\to 1} \left\langle \frac{1-t}{1-t^2}, \frac{\ln t}{1-t}, 5t\right\rangle$.

Answer: 1. $\langle 1, 5, 0\rangle$.

2. $\langle 1, 1, 0\rangle$.
3. $\left\langle \frac{1}{4}, 4, 0\right\rangle$.
4. $\left\langle \frac{1}{2}, -1, 5\right\rangle$.

9. (§13.2) Given $\boldsymbol{R}(t)$, find its antiderivative $\int \boldsymbol{R}(t)\,\mathrm{d}t$.

1. $\boldsymbol{R}(t) = \left\langle 2t - 1, t^2, \cos t\right\rangle$.
2. $\boldsymbol{R}(t) = \left\langle \frac{1}{1+t^2}, \frac{1}{t}, \sqrt{2t + 1}\right\rangle$.

Answer: 1. $\int \boldsymbol{R}(t)\,\mathrm{d}t = \left\langle t^2 - t, t^3/3, \sin t\right\rangle + \boldsymbol{C}$

2. $\int \boldsymbol{R}(t)\,\mathrm{d}t = \left\langle \arctan t, \ln t, (2t+1)^{3/2}/3 \right\rangle + \boldsymbol{C}$

10. (§13.2) Given vector valued functions $\boldsymbol{R}(t)$ and $\boldsymbol{S}(t)$, which describe the position of two particles, find whether the particles collide. Find if the paths of the two particles intersect.

1. $\boldsymbol{R}(t) = \left\langle 2t, t^2, \cos t \right\rangle$, $\boldsymbol{S}(t) = \left\langle \sin t, -e^t, 4t \right\rangle$.
2. $\boldsymbol{R}(t) = \langle 1+3t, 2-5t, -2+t \rangle$, $\boldsymbol{S}(t) = \langle 5-t, 1-4t, -3+2t \rangle$.
3. $\boldsymbol{R}(t) = \langle 1+t, 2+t, 3+t \rangle$, $\boldsymbol{S}(t) = \left\langle t-1, t^2, t+1 \right\rangle$.

Answer: The particles collide if there is a t such that $\boldsymbol{R}(t) = \boldsymbol{S}(t)$. The paths intersect if there are t_1, t_2 such that $\boldsymbol{R}(t_1) = \boldsymbol{S}(t_2)$.

1. These particles cannot collide, and their paths cannot cross because the y component of $\boldsymbol{R}$ is nonnegative, while the y component of $\boldsymbol{S}$ is negative.
2. Try to solve $\langle 1+3t, 2-5t, -2+t \rangle = \langle 5-t, 1-4t, -3+2t \rangle$. This is three equations with one unknown, an overdetermined systems. Solving the first equation gives $1+3t = 5-t$, which has solution $t = 1$. This value of t satisfies the other two equations, as you can verify:

$$\boldsymbol{R}(1) = \langle 4, -3, -1 \rangle = \boldsymbol{S}(1).$$

3. First try to solve $\langle 1+t, 2+t, 3+t \rangle = \left\langle t-1, t^2, t+1 \right\rangle$. This is three equations with one unknown. Solving the first equation gives $1+t = t-1$. This has no solution. Thus we should look for path intersection. So we try to solve $\langle 1+t_1, 2+t_1, 3+t_1 \rangle = \left\langle t_2-1, t_2^2, t_2+1 \right\rangle$. This is three equations with two unknowns, which is still overdetermined, but may have a solution Solving the first equation gives $1+t_1 = t_2-1$, or $t_1+2 = t_2$. Plugging this into the second equation gives $2+t_1 = t_2^2 = (t_1+2)^2 = t_1^2+4t_1+4$. This is the quadratic equation $t_1^2+3t_1+2 = 0$, with solutions $t_1 = -1, -2$. This gives prospective solutions $t_1 = -1, t_2 = 1$, and $t_1 = -2, t_2 = 0$. Check if these satisfy the third equation $3+t_1 = t_2+1$. They both satisfy it. So we get two points of intersection:

$$\boldsymbol{R}(-1) = \langle 0, 1, 2 \rangle = \boldsymbol{S}(1), \quad \text{and} \quad \boldsymbol{R}(-2) = \langle -1, 0, 1 \rangle = \boldsymbol{S}(0).$$

11. (§13.2) Find $\boldsymbol{R}'(t)$ for

1. $\boldsymbol{R}(t) = \langle t, 3-2t, 4+6t \rangle$.
2. $\boldsymbol{R}(t) = \left\langle t+1, 5, t^2 \right\rangle$.
3. $\boldsymbol{R}(t) = \langle 4, 7, 1 \rangle \times \left\langle t+1, 5, t^2 \right\rangle$.
4. $\boldsymbol{R}(t) = \langle \arctan t, \sin t, \cos t \rangle$.
5. $\boldsymbol{R}(t) = \left\langle \arctan\left(e^t\right), \sin\left(e^t\right), \cos\left(e^t\right) \right\rangle$.
6. $\boldsymbol{R}(t) = \langle \cos t, \sin t, 3t \rangle$.
7. $\boldsymbol{R}(t) = \left\langle (t+1)\sin t, 5\sin t, t^2 \sin t \right\rangle$.

Answer: 1. $\boldsymbol{R}'(t) = \langle 1, -2, 6 \rangle$.

2. $\boldsymbol{R}'(t) = \langle 1, 0, 2t \rangle$.
3. $\boldsymbol{R}'(t) = \langle 4, 7, 1 \rangle \times \langle 1, 0, 2t \rangle = \langle 14t, 1-8t, -7 \rangle$.

4. $\boldsymbol{R}'(t) = \left\langle \frac{1}{1+t^2}, \cos t, -\sin t \right\rangle$.
5. $\boldsymbol{R}'(t) = \langle -\sin t, \cos t, 3 \rangle$.
6. $\boldsymbol{R}'(t) = \sin t \langle 1, 0, 2t \rangle + \cos t \left\langle t+1, 5, t^2 \right\rangle$

12. (§13.3) Given vector valued function $\boldsymbol{R}(t)$, find the arc length of the curve traced by $\boldsymbol{R}$ for t between t_0 and t_1, *e.g.*,

1. $\boldsymbol{R}(t) = \langle 1+3t, 3, 1-4t \rangle$, $t_0 = 0$, $t_1 = 3$.
2. $\boldsymbol{R}(t) = \langle 2\cos t, 2\sin t \rangle, 0 \le t \le \pi$.
3. $\boldsymbol{R}(t) = \langle \cos t, \sin t, 3t \rangle$, $t_0 = 0$, $t_1 = 4\pi$.
4. $\boldsymbol{R}(t) = \left\langle t/\sqrt{2}, t/\sqrt{2}, \ln(\sec t) \right\rangle$, $t_0 = 0$, $t_1 = \pi/3$.
5. $\boldsymbol{R}(t) = \left\langle t, t^3/3 + 1/(4t) \right\rangle$, $t_0 = 1$, $t_1 = 2$.
6. $\boldsymbol{R}(t) = \left\langle \sqrt{2}t, e^t, e^{-t} \right\rangle, 0 \le t \le 1$.
7. $\boldsymbol{R}(t) = \left\langle t^2, t, 3 \right\rangle, 0 \le t \le 1$.

Answer: Solveable problems of this type are few and far between. These examples pretty much exhaust my supply of good problems. Recall that the arc length of $\boldsymbol{c}(t)$ for $t_0 \le t \le t_1$ is

$$\int_{t_0}^{t_1} \left|\boldsymbol{c}'(t)\right| \, \mathrm{d}t$$

1. $\int_0^3 5 \, \mathrm{d}t = 15$
2. 2π.
3. $\int_0^{4\pi} \sqrt{10} \, \mathrm{d}t = 4\pi\sqrt{10}$.
4. $\int_0^{\pi/3} \sec t \, \mathrm{d}t = \ln(\sec t + \cos t)\Big|_0^{\pi/3} = 2 + \sqrt{3} - 1 = 1 + \sqrt{3}$.
5. We compute $\boldsymbol{R}'(t) = \left\langle 1, t^2 - 1/(4t^2) \right\rangle$. Now note that

$$\begin{aligned} \left|\boldsymbol{R}'(t)\right| &= \sqrt{1 + (t^2 - 1/(4t^2))^2} = \sqrt{1 + t^4 - 1/2 + 1/(16t^4)} \\ &= \sqrt{t^4 + 1/2 + 1/(16t^4)} = \sqrt{(t^2 + 1/(4t^2))^2} = t^2 + 1/(4t^2) \end{aligned}$$

   Thus the answer is

$$\int_1^2 t^2 + 1/(4t^2) \, \mathrm{d}t = t^3/3 - 1/(4t)\Big|_1^2 = 8/3 - 1/8 - (1/3 - 1/4) = 7/3 + 1/8$$

6. $e^1 - e^{-1}$.
7. The integral might be tricky. I get $\frac{\sqrt{5}}{2} + \frac{1}{4} \operatorname{arcsinh} 2$.

13. (§13.4) Given the position of a particle, $\boldsymbol{R}(t)$, find its velocity, speed, and acceleration. *e.g.*,:

1. $\boldsymbol{R}(t) = \langle 2\cos t, 2\sin t \rangle$.
2. $\boldsymbol{R}(t) = \left\langle \sqrt{2}t, e^t, e^{-t} \right\rangle$.

3. $\boldsymbol{R}(t) = \left\langle t^2, t, 3 \right\rangle$.

Answer: 1. $\boldsymbol{v}(t) = \langle -2\sin t, 2\cos t \rangle$, $s(t) = 2$, $\boldsymbol{a}(t) = \langle -2\cos t, -2\sin t \rangle$.

2. $\boldsymbol{v}(t) = \left\langle \sqrt{2}, e^t, -e^{-t} \right\rangle$, $s(t) = e^t + e^{-t}$, $\boldsymbol{a}(t) = \left\langle 0, e^t, e^{-t} \right\rangle$.

3. $\boldsymbol{v}(t) = \langle 2t, 1, 0 \rangle$, $s(t) = \sqrt{4t^2 + 1}$, $\boldsymbol{a}(t) = \langle 2, 0, 0 \rangle$.

14. (§13.4) Given the acceleration of a particle and its initial position and velocity, find the position of the particle. *e.g.*,:

1. $\boldsymbol{a}(t) = \langle 1, t, \sin t \rangle$, $\boldsymbol{v}(0) = \langle 0, 1, 0 \rangle$, $\boldsymbol{R}(0) = \langle 0, 0, 0 \rangle$.
2. $\boldsymbol{a}(t) = \langle \sin t, \cos t \rangle$, $\boldsymbol{v}(0) = \langle 1, 0 \rangle$, $\boldsymbol{R}(0) = \langle 0, 1 \rangle$.

Answer: 1. $\boldsymbol{R}(t) = \left\langle \frac{1}{2}t^2, \frac{1}{6}t^3 + t, t - \sin t \right\rangle$.

2. $\boldsymbol{R}(t) = \langle 2t - \sin t, 2 - \cos t \rangle$.

15. (§14.1) You should have some basic understanding of functions of two or more variables. *e.g.*,: given $f(x, y)$, be able to evaluate $f(1, 2)$; find the domain of f; sketch a graph of $f(x, y)$, or the contours (level sets) of $f(x, y)$.

1. What is the domain of $f(x, y) = \frac{\sqrt{x-y}}{y}$?
2. Graph the level sets of $f(x, y) = \frac{1}{4}x^2 + y^2$.
3. Graph the function $f(x, y) = \sqrt{x^2 + y^2}$.
4. Graph the function and level sets of $f(x, y) = 3x - 4y + 2$.

Answer: 1. The points (x, y) with $x \geq y$, and $y \neq 0$.

2. These are ellipses.
3. A paraboloid.
4. The graph is the plane $-3x + 4y + z = 2$, the level sets are lines.